Infinite dimensional analysis and the Chern-Simons path integral

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Abstract

Using the framework of White Noise Analysis we give a rigorous implementation of the gauge fixed Chern-Simons path integral associated to an arbitrary simple simply-connected compact structure group G and a simple class of (ribbon) links L in the base manifold $M = S^2 \times S^1$.

Key words: Chern-Simons theory, Feynman path integrals, quantum invariants, shadow invariant, White Noise Analysis

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1 Introduction

In the 1980s and the beginning of the 1990s Jones, Witten, Turaev, Reshetikhin, Kontsevich, and others revolutionized Knot Theory and created a whole new area which is now called "Quantum Topology". Quantum Topology is both a deep and a beautiful theory, beautiful in the sense that it naturally connects a large number of branches of Mathematics and Physics[1] like Algebra (Lie algebras, affine Lie algebras, quantum groups, ..., and the corresponding representation theories), low-dimensional Topology & Knot Theory, Riemannian Geometry & Global Analysis, Infinite Dimensional Analysis, and Quantum Field Theory (Gauge Field Theory, Conformal Field Theory, Quantum Gravity, String Theory).

The first major step towards Quantum Topology was the discovery of the Jones polynomial and its generalizations (in particular, the HOMFLY and the Kauffman polynomials) in 1984 and 1985. In 1988, Witten demonstrated in a celebrated paper [29] that the heuristic Feynman path integral associated to a certain 3-dimensional gauge field theory can be used to give a very elegant and intrinsically 3-dimensional "definition" of the Jones polynomial and the other knot polynomials mentioned above. The aforementioned gauge theory is the so-called (pure) "Chern-Simons model" which is specified by a triple (M, G, k) where M is an oriented connected 3-dimensional manifold (usually compact), G is a semi-simple Lie group (often compact and simply-connected), and $k \in \mathbb{N}$ is a fixed parameter ("the level"). In the very important special

[1] this is reflected also in the present paper: Quantum Gauge Field Theory plays the main role in Sec. 2.2 and Sec. 2.3, Riemannian Geometry/Global Analysis in Sec. 2.5, Infinite Dimensional Analysis in Sec. 3, and Algebra and low-dimensional Topology in Appendix A

case $G = SU(N)$, $N \geq 2$, the action function S_{CS} of the Chern-Simons model is given explicitly by

$$S_{CS}(A) = k \int_M \mathrm{Tr}(A \wedge dA + \tfrac{2}{3} A \wedge A \wedge A), \qquad A \in \mathcal{A} \tag{1.1}$$

where $\mathcal{A}$ is the space of all $su(N)$-valued 1-forms A on M, $\wedge$ the wedge product associated to the multiplication of $\mathrm{Mat}(N, \mathbb{C})$, and $\mathrm{Tr} : \mathrm{Mat}(N, \mathbb{C}) \to \mathbb{C}$ the suitably normalized trace.

The Chern-Simons "path space measure" is the informal complex measure $\exp(iS_{CS}(A))DA$ where DA is the (ill-defined) Lebesgue measure on the infinite dimensional space $\mathcal{A}$. The Chern-Simons path integral (functional) $\int \cdots \exp(iS_{CS}(A))DA$ not only gives a unifying framework for the aforementioned knot polynomials[2] but also leads naturally to a generalization of these invariants to all closed 3-manifolds, the "Jones-Witten invariants".

Using several heuristic arguments, some of them from Conformal Field Theory, Witten was able to evaluate the Jones-Witten invariants explicitly. Two years later Reshetikhin and Turaev finally found a rigorous (and equivalent) version of the Jones-Witten invariants, the so-called "Reshetikhin-Turaev invariants", [24, 23]. The approach in [24, 23] is algebraic and very different from the path integral approach. In particular, it is based on the representation theory of quantum groups and "surgery operations" on the relevant 3-manifolds. Turaev [27] later found an equivalent approach called the "shadow world" approach, which is also based on quantum group representations but uses certain finite "state sums" instead of the surgery operations.

Many open questions in the field of Quantum Topology are closely related to the following problem which is generally considered to be one of the major open problems in the field (cf. [19] und [25]):

(P1) Find a rigorous realization[3] of the (original or gauge fixed) Chern-Simons path integral expressions for all simply-connected compact Lie groups G and all (compact oriented 3-dimensional) base manifolds M.

Since (P1) seems to be a very hard problem it makes sense to restrict oneself first to the following weakened version of (P1).

(P1)' Find a rigorous realization of the (suitably gauge fixed) Chern-Simons path integral expressions for all compact and simply-connected Lie groups G and some fixed base manifold M like, e.g., $M = S^3$ or $M = S^2 \times S^1$.

Note that the quadratic part of the Chern-Simons action function S_{CS} is degenerate. Moreover, there is a cubic term, which is clearly not semi-bounded. Because of this we cannot hope to be able to transform the (original or gauge-fixed) Chern-Simons path space measure into a bounded positive (heuristic) measure by applying a so-called "Wick rotation"[4]. This means that the well-established theory of positive bounded measures on infinite dimensional topological vector spaces is not as useful as in standard Constructive QFT. Instead of working with bounded measures one can try to work with a suitable notion of distributions on infinite dimensional spaces as provided, e.g. by White Noise Analysis.

And indeed, as we will show in the present paper, White Noise Analysis can be used to make progress towards the solution of Problem (P1)'.

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[2] the polynomial invariants mentioned above are given by $\mathrm{WLO}(L)$ as defined in Eq. (2.5) below. In the special case $M = S^3$, $G = SU(2)$, and where each representation ρ_i appearing in in Eq. (2.5) is the fundamental representation of $SU(2)$ $\mathrm{WLO}(L)$ is given by an explicit expression involving the Jones polynomial of L.

[3] here the word "realization" is meant to imply that the values of the rigorously defined Chern-Simons path integral expressions coincide with the corresponding Reshetikhin-Turaev invariant

[4] this is different from the situation in many other bosonic QFTs where the path space measure can be transformed into a bounded positive (heuristic) measure by means of a Wick rotation

More precisely, for the special manifold $M = S^2 \times S^1$ we will give a rigorous realization of Witten's path integrals (after a suitable gauge fixing has been applied). We expect (cf. Conjecture 2 below) that at least for a simple type of (ribbon) links L the aforementioned rigorous realization reproduces Turaev's shadow invariant.

Our paper is based on the "torus gauge fixing" approach to Chern-Simons theory on base manifolds of the form $M = \Sigma \times S^1$ which was developed in [2, 3, 4, 8, 9, 10, 6] (see also [12] for later developments). In [8, 10, 11] we sketched how a fully rigorous realization of the torus gauge fixed Chern-Simons path integral could be obtained within the framework of White Noise Analysis (using ideas from [20, 1, 7]). We remark here, however, that several constructions in [8, 10, 11] (and also in [6]) are unnecessarily complicated, which is why in the present paper we will reconsider the issue and make the following changes and improvements:

1. Instead of the heuristic formula Eq. (15) in [11], which was used as the starting point for the treatment in [11], we will use the simplified and more natural heuristic formula Eq. (2.7) below (which was first derived in [12]). In particular, the space $\hat{\mathcal{A}}^{\perp}$ appearing in [11] is now replaced by the space $\check{\mathcal{A}}^{\perp}$. Moreover, the singular 1-forms $A^{\perp}_{sing}(\mathrm{h})$ appearing in Eq. (15) in [11] are now absent.

2. We use an alternative rigorous implementation of the heuristic expression $\mathrm{Det}(B) := \mathrm{Det}_{FP}(B)\check{Z}(B)$ in Sec. 2.5, cf. Eqs. (2.21) below. The implementation in Sec. 2.5 is considerably more natural than the "ad hoc ansatz" in Eq. (13) in [11] for $\mathrm{Det}(B)$.

3. The framing procedure is implemented in a different way. We no longer use a family of diffeomorphisms $(\phi_s)_{s>0}$ of $M = \Sigma \times S^1$ in order to deform the two Hida distributions appearing in [11]. Instead we will work with the undeformed Hida distributions but replace the (smeared) loops by (smeared) ribbons. This has two important advantages: Firstly, it makes the alternative, natural definition of $\mathrm{Det}(B)$ mentioned above possible. Secondly, it seems to eliminate the "loop smearing dependence problem" which would probably have appeared by carrying out the original approach in [11].

We remark that in [12, 13] we have developed an alternative "simplicial" approach for making sense of the RHS of Eq. (2.7), see Sec. 4.2 below for a brief comparison of the rigorous continuum approach of the present paper and the simplicial approach in [12, 13].

The present paper is organized as follows:

In Secs 2.1–2.3 we recall the relevant heuristic formulas for the Wilson loop observables in Chern-Simons theory, first the original formula Eq. (2.5) and later the modified formula which was obtained in [12] by applying torus gauge fixing in the special case where $M = \Sigma \times S^1$, cf. Eq. (2.7). In Sec. 2.4 we introduce "infinitesimal ribbons" and rewrite the heuristic formula Eq (2.7), obtaining Eq. (2.17). In Sec. 2.5 we give the rigorous definition of the expression $\mathrm{Det}(B) = \mathrm{Det}_{FP}(B)\check{Z}(B)$ mentioned above. In Sec. 2.6 we finally rewrite the heuristic formula Eq. (2.17) obtained in Sec. 2.4 in a suitable way (for the special case $\Sigma = S^2$) arriving at the heuristic formula Eq. (2.44).

In Sec. 3 we then explain how one can make rigorous sense of the RHS of the aforementioned Eq. (2.44). In Sec. 4 we conclude the main part of this paper with a brief discussion of our results.

In Appendix A we briefly recall the definition of Turaev's shadow invariant in the special case relevant for us, i.e. $M = S^2 \times S^1$. In Appendix B we fill in some technical details which were omitted in Sec. 3.

2 The heuristic Chern-Simons path integral in the torus gauge

We fix a simple simply-connected compact Lie group G and a maximal torus T of G. By $\mathfrak{g}$ and $\mathfrak{t}$ we will denote the Lie algebras of G and T and by $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ or simply by $\langle \cdot, \cdot \rangle$ the unique

Ad-invariant scalar product on $\mathfrak{g}$ satisfying the normalization condition $\langle \check{\alpha}, \check{\alpha} \rangle = 2$ for every short coroot $\check{\alpha}$ w.r.t. $(\mathfrak{g}, \mathfrak{t})$.

Moreover, we will fix a compact oriented 3-manifold M of the form $M = \Sigma \times S^1$ where Σ is a compact oriented surface. Finally, we fix an (ordered and oriented) "link" L in M, i.e. a finite tuple $L = (l_1, \ldots, l_m)$, $m \in \mathbb{N}$, of pairwise non-intersecting knots l_i and we equip each l_i with a "color", i.e. a finite-dimensional complex representation ρ_i of G. Recall that a "knot" in M is an embedding $l : S^1 \to M$. Using the surjection $[0,1] \ni t \mapsto e^{2\pi i t} \in S^1 \cong \{z \in \mathbb{C} \mid |z| = 1\}$ we can consider each knot as a loop $l : [0,1] \to M$, $l(0) = l(1)$, in the obvious way.

2.1 Basic spaces

As in [12] we will use the following notation[5]

$$\mathcal{B} = C^\infty(\Sigma, \mathfrak{t}) \tag{2.1a}$$
$$\mathcal{A} = \Omega^1(M, \mathfrak{g}) \tag{2.1b}$$
$$\mathcal{A}_\Sigma = \Omega^1(\Sigma, \mathfrak{g}) \tag{2.1c}$$
$$\mathcal{A}_{\Sigma,\mathfrak{t}} = \Omega^1(\Sigma, \mathfrak{t}), \quad \mathcal{A}_{\Sigma,\mathfrak{k}} = \Omega^1(\Sigma, \mathfrak{k}) \tag{2.1d}$$
$$\mathcal{A}^\perp = \{A \in \mathcal{A} \mid A(\partial/\partial t) = 0\} \tag{2.1e}$$
$$\check{\mathcal{A}}^\perp = \{A^\perp \in \mathcal{A}^\perp \mid \int A^\perp(t) dt \in \mathcal{A}_{\Sigma,\mathfrak{k}}\} \tag{2.1f}$$
$$\mathcal{A}^\perp_c = \{A^\perp \in \mathcal{A}^\perp \mid \ A^\perp \text{ is constant and } \mathcal{A}_{\Sigma,\mathfrak{t}}\text{-valued}\} \tag{2.1g}$$

Here $\mathfrak{k}$ is the orthogonal complement of $\mathfrak{t}$ in $\mathfrak{g}$ w.r.t. $\langle \cdot, \cdot \rangle$, dt is the normalized translation-invariant volume form on S^1, and $\partial/\partial t$ is the vector field on $M = \Sigma \times S^1$ obtained by "lifting" in the obvious way the normalized translation-invariant vector field $\partial/\partial t$ on S^1. Moreover, in Eqs. (2.1f) and (2.1g) we used the "obvious" identification (cf. Sec. 2.3.1 in [12])

$$\mathcal{A}^\perp \cong C^\infty(S^1, \mathcal{A}_\Sigma) \tag{2.2}$$

where $C^\infty(S^1, \mathcal{A}_\Sigma)$ is the space of maps $f : S^1 \to \mathcal{A}_\Sigma$ which are "smooth" in the sense that $\Sigma \times S^1 \ni (\sigma, t) \mapsto (f(t))(X_\sigma) \in \mathfrak{g}$ is smooth for every smooth vector field X on Σ. It follows from the definitions above that

$$\mathcal{A}^\perp = \check{\mathcal{A}}^\perp \oplus \mathcal{A}^\perp_c \tag{2.3}$$

2.2 The heuristic Wilson loop observables

The Chern-Simons action function $S_{CS} : \mathcal{A} \to \mathbb{R}$ associated to M, G, and the "level" $k \in \mathbb{Z}\backslash\{0\}$ is given by[6]

$$S_{CS}(A) = -k\pi \int_M \langle A \wedge dA \rangle + \tfrac{1}{3} \langle A \wedge [A \wedge A] \rangle, \quad A \in \mathcal{A} \tag{2.4}$$

where $[\cdot \wedge \cdot]$ denotes the wedge product associated to the Lie bracket $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ and where $\langle \cdot \wedge \cdot \rangle$ denotes the wedge product associated to the scalar product $\langle \cdot, \cdot \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$.

Recall that the heuristic Wilson loop observable $\mathrm{WLO}(L)$ of a link $L = (l_1, l_2, \ldots, l_m)$ in M with "colors" $(\rho_1, \rho_2, \ldots, \rho_m)$ is given by the informal "path integral" expression

$$\mathrm{WLO}(L) := \int_{\mathcal{A}} \prod\nolimits_i \mathrm{Tr}_{\rho_i}(\mathrm{Hol}_{l_i}(A)) \exp(i S_{CS}(A)) DA \tag{2.5}$$

[5] Here $\Omega^p(N, V)$ denotes the space of V-valued p-forms on a smooth manifold N

[6] Eq. (2.4) generalizes Eq. (1.1) in Sec. 1. In Eq. (1.1) the factor $-\pi$ is hidden in the trace functional $\mathrm{Tr} : \mathrm{Mat}(N, \mathbb{C}) \to \mathbb{C}$. Moreover, in Eq. (1.1) we have a factor 2/3 instead of 1/3 because the wedge product $\wedge$ in Eq. (1.1) differs from each of the two wedge products in Eq. (2.4)

where $\mathrm{Hol}_l(A) \in G$ is the holonomy of $A \in \mathcal{A}$ around the loop $l \in \{l_1, \ldots, l_m\}$ and where DA is the (ill-defined) "Lebesgue measure" on the infinite-dimensional space $\mathcal{A}$. We will use the following explicit formula for $\mathrm{Hol}_l(A)$

$$\mathrm{Hol}_l(A) = \lim_{n\to\infty} \prod_{j=1}^{n} \exp\big(\tfrac{1}{n} A(l'(t))\big)_{|t=j/n} \tag{2.6}$$

where $\exp : \mathfrak{g} \to G$ is the exponential map of G.

2.3 The basic heuristic formula from [12]

The starting point for the main part of [12] was a second heuristic formula for $\mathrm{WLO}(L)$ which one obtains from Eq. (2.5) above after applying a suitable gauge fixing.

Let $\pi_\Sigma : \Sigma \times S^1 \to \Sigma$ be the canonical projection. For each loop l_i appearing in the link L we set $l^i_\Sigma := \pi_\Sigma \circ l_i$. Moreover, we fix $\sigma_0 \in \Sigma$ such that

$$\sigma_0 \notin \mathrm{arc}(L_\Sigma) := \bigcup_i \mathrm{arc}(l^i_\Sigma)$$

By applying "abstract torus gauge fixing" (cf. Sec. 2.2.4 in [12]) and a suitable change of variable one can derive at a heuristic level (cf. Eq. (2.53) in [12])

$$\begin{aligned}\mathrm{WLO}(L) \sim \sum_{y\in I} \int_{\mathcal{A}^\perp_c \times \mathcal{B}} \Big\{ & 1_{\mathcal{B}_{reg}}(B)\, \mathrm{Det}_{FP}(B) \\ & \times \Big[\int_{\check{\mathcal{A}}^\perp} \prod_i \mathrm{Tr}_{\rho_i}\big(\mathrm{Hol}_{l_i}(\check{A}^\perp + A^\perp_c, B)\big) \exp(iS_{CS}(\check{A}^\perp, B)) D\check{A}^\perp \Big] \\ & \times \exp\big(-2\pi i k \langle y, B(\sigma_0)\rangle\big) \Big\} \exp(iS_{CS}(A^\perp_c, B))(DA^\perp_c \otimes DB)\end{aligned} \tag{2.7}$$

where "$\sim$" denotes equality up to a multiplicative "constant"[7] C, where $I := \ker(\exp_{|\mathfrak{t}}) \subset \mathfrak{t}$, where DB and $DA^\perp_c$ are the informal "Lebesgue measures" on the infinite-dimensional spaces $\mathcal{B}$ and $\mathcal{A}^\perp_c$, and where have set

$$\mathcal{B}_{reg} := \{B \in \mathcal{B} \mid \forall \sigma \in \Sigma : B(\sigma) \in \mathfrak{t}_{reg}\} = C^\infty(\Sigma, \mathfrak{t}_{reg}) \tag{2.8}$$

with $\mathfrak{t}_{reg} := \exp^{-1}(T_{reg})$, T_{reg} being the set of "regular" elements[8] of T.

Moreover, we have set for each $B \in \mathcal{B}$, $A^\perp \in \mathcal{A}^\perp$

$$S_{CS}(A^\perp, B) := S_{CS}(A^\perp + Bdt) \tag{2.9}$$

$$\mathrm{Hol}_l(A^\perp, B) := \mathrm{Hol}_l(A^\perp + Bdt) \tag{2.10}$$

Here dt is the real-valued 1-form on $M = \Sigma \times S^1$ obtained by pulling back the 1-form dt on S^1 by means of the canonical projection $\pi_{S^1} : \Sigma \times S^1 \to S^1$. Finally, $\mathrm{Det}_{FP}(B)$ is the informal expression given by

$$\mathrm{Det}_{FP}(B) := \det\big(1_{\mathfrak{k}} - \exp(\mathrm{ad}(B))_{|\mathfrak{k}}\big) \tag{2.11}$$

where $1_{\mathfrak{k}} - \exp(\mathrm{ad}(B))_{|\mathfrak{k}}$ is the linear operator on $C^\infty(\Sigma, \mathfrak{k})$ given by

$$(1_{\mathfrak{k}} - \exp(\mathrm{ad}(B))_{|\mathfrak{k}} \cdot f)(\sigma) = (\mathrm{id}_{\mathfrak{k}} - \exp(\mathrm{ad}(B(\sigma)))_{|\mathfrak{k}}) \cdot f(\sigma) \qquad \forall \sigma \in \Sigma, \quad \forall f \in C^\infty(\Sigma, \mathfrak{k})$$

where on the RHS $\mathrm{id}_{\mathfrak{k}}$ is the identity on $\mathfrak{k}$.

For the rest of this paper we will now fix an auxiliary Riemannian metric $\mathbf{g} = \mathbf{g}_\Sigma$ on Σ. After doing so we obtain scalar products $\ll \cdot, \cdot \gg_{\mathcal{A}_\Sigma}$ and $\ll \cdot, \cdot \gg_{\mathcal{A}^\perp}$ on $\mathcal{A}_\Sigma$ and $\mathcal{A}^\perp \cong C^\infty(S^1, \mathcal{A}_\Sigma)$ in a

[7] "constant" in the sense that C does not depend on L. By contrast, C may depend on G, Σ, and k.

[8] i.e. the set of all $t \in T$ which are not contained in a different maximal torus T'

natural way. Moreover, we obtain a well-defined Hodge star operator $\star : \mathcal{A}_\Sigma \to \mathcal{A}_\Sigma$ which induces an operator $\star : C^\infty(S^1, \mathcal{A}_\Sigma) \to C^\infty(S^1, \mathcal{A}_\Sigma)$ in the obvious way, i.e. by $(\star A^\perp)(t) = \star(A^\perp(t))$ for all $A^\perp \in \mathcal{A}^\perp$ and $t \in S^1$. We have the following explicit formula (cf. Eq. (2.48) in [12])

$$S_{CS}(A^\perp, B) = \pi k \ll A^\perp, \star\big(\tfrac{\partial}{\partial t} + \mathrm{ad}(B)\big)A^\perp \gg_{\mathcal{A}^\perp} + 2\pi k \ll \star A^\perp, dB \gg_{\mathcal{A}^\perp} \tag{2.12}$$

for all $B \in \mathcal{B}$ and $A^\perp \in \mathcal{A}^\perp$, which implies

$$S_{CS}(\check{A}^\perp, B) = \pi k \ll \check{A}^\perp, \star\big(\tfrac{\partial}{\partial t} + \mathrm{ad}(B)\big)\check{A}^\perp \gg_{\mathcal{A}^\perp} \tag{2.13}$$

$$S_{CS}(A_c^\perp, B) = 2\pi k \ll \star A_c^\perp, dB \gg_{\mathcal{A}^\perp} \tag{2.14}$$

for $B \in \mathcal{B}$, $\check{A}^\perp \in \check{\mathcal{A}}^\perp$, and $A_c^\perp \in \mathcal{A}_c^\perp$.

The following informal definitions will be useful in Sec. 2.4 below:
For each $B \in \mathcal{B}$ we set

$$\check{Z}(B) := \int \exp(iS_{CS}(\check{A}^\perp, B)) D\check{A}^\perp, \tag{2.15}$$

$$d\mu_B^\perp := \tfrac{1}{\check{Z}(B)} \exp(iS_{CS}(\check{A}^\perp, B)) D\check{A}^\perp \tag{2.16}$$

Moreover, we will denote by $d_{\mathbf{g}}$ the distance function on Σ and by $d\mu_{\mathbf{g}}$ the volume measure on Σ which are associated to the Riemannian metric $\mathbf{g}$.

2.4 Ribbon version of Eq. (2.7)

It is well-known in the mathematics and physics literature on quantum 3-manifold invariants that rather than working with links one actually has to work with framed links or, equivalently, with ribbon links (see below) if one wants to get meaningful results.

From the knot theory point of view the framed link picture and the ribbon link picture are equivalent. However, the ribbon picture seems to be better suited for the study of the Chern-Simons path integral in the torus gauge.

For every (closed) ribbon R in $\Sigma \times S^1$, i.e. every smooth embedding $R : S^1 \times [0,1] \to \Sigma \times S^1$, we define

$$\mathrm{Hol}_R(A) := \lim_{n\to\infty} \prod_{j=1}^n \exp\big(\tfrac{1}{n}\int_0^1 A(R_u'(t))du\big)_{|t=j/n} \in G$$

where R_u, for $u \in [0,1]$, is the loop in $\Sigma \times S^1$ given by $R_u(t) = R(t,u)$ for all $t \in S^1$.

A ribbon link in $\Sigma \times S^1$ is a finite tuple of non-intersecting closed ribbons in in $\Sigma \times S^1$. We will replace the link $L = (l_1, l_2, \ldots, l_m)$ by a ribbon link $L_{ribb} = (R_1, R_2, \ldots, R_m)$ where each R_i, $i \le m$, is chosen such that $l_i(t) = R_i(t, 1/2)$ for all $t \in S^1$.

Moreover, we "scale" each R_i, i.e. for each $s \in (0,1)$ we introduce the ribbon $R_i^{(s)}$ by $R_i^{(s)}(t,u) := R_i(t, s\cdot(u - 1/2) + 1/2)$ for all $t \in S^1$ and $u \in [0,1]$.

Then we replace $\mathrm{Hol}_{l_i}(A)$ appearing in Eq. (2.7) (with $A = \check{A}^\perp + A_c^\perp + Bdt$) by $\mathrm{Hol}_{R_i^{(s)}}(A)$. Moreover, we include a $s \to 0$ limit.

Convention 1 *We will usually write simply L instead of L_{ribb} when no confusion can arise.*

Remark 2.1 *The inclusion*[9] *of the limit $s \to 0$ above is the formal implementation of the intuitive idea that our ribbons should have "infinitesimal width".*

[9]if one does not include this limit $s \to 0$ it may still be possible to derive a result like Eq. (3.22) below for ribbon links L fulfilling Assumption 1 but most probably not for general ribbon links.

After these preparations we arrive at the following ribbon analogue of Eq. (2.7) above

$$\begin{aligned} \mathrm{WLO}(L) \sim \lim_{s\to 0} \sum\nolimits_{y\in I} \int_{\mathcal{A}_c^\perp \times \mathcal{B}} \Big\{ & 1_{\mathcal{B}_{reg}}(B) \, \mathrm{Det}_{FP}(B) \check{Z}(B) \\ & \times \Big[\int_{\check{\mathcal{A}}^\perp} \prod\nolimits_i \mathrm{Tr}_{\rho_i}\big(\mathrm{Hol}_{R_i^{(s)}}(\check{A}^\perp, A_c^\perp, B)\big) d\mu_B^\perp(\check{A}^\perp) \Big] \\ & \times \exp\big(-2\pi i k \langle y, B(\sigma_0)\rangle\big) \Big\} \exp(iS_{CS}(A_c^\perp, B))(DA_c^\perp \otimes DB) \end{aligned} \quad (2.17)$$

where we have set (cf. Sec. 2.3 above)

$$\mathrm{Hol}_{R_i^{(s)}}(\check{A}^\perp, A_c^\perp, B) := \mathrm{Hol}_{R_i^{(s)}}(\check{A}^\perp + A_c^\perp + B dt)$$

In the following we set

$$R_\Sigma^i := \pi_\Sigma \circ R_i$$

From now on we will restrict ourselves to ribbon links $L = (R_1, R_2, \ldots, R_m)$ fulfilling the following assumption.

Assumption 1 *The maps R_Σ^i, $i \le m$, neither intersect themselves nor each other. More precisely: Each R_Σ^i, $i \le m$, is an injection $S^1 \times [0,1] \to \Sigma$ and we have $\mathrm{Image}(R_\Sigma^i) \cap \mathrm{Image}(R_\Sigma^j) = \emptyset$ if $i \neq j$.*

Remark 2.2 *It is interesting to consider also the weakened version of Assumption 1 where instead of demanding that for each $i \le m$ the map $R_\Sigma^i : S^1 \times [0,1] \to \Sigma$ is an injection we only demand that $R_\Sigma^i(t_1, u_1) \neq R_\Sigma^i(t_2, u_2)$ for all $t_1, t_2 \in S^1$ and all $u_1, u_2 \in [0,1]$ fulfilling $u_1 \neq u_2$ and that for fixed $u \in [0,1]$ the Image of $S^1 \ni t \mapsto R_\Sigma^i(t, u) \in \Sigma$ lies on an embedded circle in Σ.*

Observe that this includes a certain class of torus (ribbon) knots. I expect that the obvious generalization of Conjecture 1 below will also hold for the aforementioned weakened version of Assumption 1 (if combined with a suitably modified version of Assumption 2 below). Moreover, I expect that Conjecture 2 below can be generalized to this more general situation and that by doing so one can obtain a continuum analogue of Theorem 5.7 in [14].

2.5 Definition of $\mathrm{Det}_{rig}(B)$

We will now explain how, using a suitable "heat kernel regularization", one can make rigorous sense of the expression

$$\mathrm{Det}(B) := \mathrm{Det}_{FP}(B) \check{Z}(B) \quad (2.18)$$

and how one can evaluate the rigorous version $\mathrm{Det}_{rig}(B)$ of $\mathrm{Det}(B)$ explicitly[10]. Here $B \in \mathcal{B}$ is fixed and $\mathrm{Det}_{FB}(B)$ and $\check{Z}(B)$ are given as in Eq. (2.11) and Eq. (2.15) above.

Remark 2.3 *The approach which we use here is a simplified version of the approach in Sec. 6 in [2]. The main difference is that we use the exponentials $e^{-\epsilon \triangle_i}$ of the original (="plain") Hodge Laplacians $\triangle_i$ while in Sec. 6 in [2] "covariant Hodge Laplacians" are used. The use of the covariant Hodge Laplacians produces an additional term containing the dual Coxeter number $c_\mathfrak{g}$ of $\mathfrak{g}$. The overall effect in the simple situation in [2] where only "vertical links" (see the paragraph after Remark 2.4 below) are used is a "shift" $k \to k + c_\mathfrak{g}$, in agreement with the shift predicted in Witten's original paper [29]. In the case of general links it is doubtful that the use of covariant Hodge Laplacian can produce a shift $k \to k + c_\mathfrak{g}$ in all places where this would be necessary. On the other hand, the fact that by working with the "plain" Hodge Laplacians we do not get a shift $k \to k + c_\mathfrak{g}$ should not be a cause for concern. It seems to be generally accepted nowadays that the occurrence and magnitude of the shift in k will depend on the regularization procedure and renormalization prescription which is applied (cf. Remark 3.2 in [12]).*

[10] in the special case where B is constant there is an additional way of doing so, cf. part ii) of Remark 2.4 below

Informally, we have

$$\check{Z}(B) \sim \left|\det\left(\tfrac{\partial}{\partial t} + \operatorname{ad}(B)\right)\right|^{-1/2} = \det\left(1_{\mathfrak{k}} - \exp(\operatorname{ad}(B))_{|\mathfrak{k}}\right)^{-1/2} \tag{2.19}$$

where $\frac{\partial}{\partial t} + \operatorname{ad}(B)$ is as in Eq. (2.13) above and where $1_{\mathfrak{k}} - \exp(\operatorname{ad}(B))_{|\mathfrak{k}}$ is the linear operator on $\mathcal{A}_{\Sigma,\mathfrak{k}} = \Omega^1(\Sigma, \mathfrak{k})$ given by

$$\forall \alpha \in \mathcal{A}_{\Sigma,\mathfrak{k}} : \forall \sigma \in \Sigma : \forall X_\sigma \in T_\sigma\Sigma : \qquad (1_{\mathfrak{k}} - \exp(\operatorname{ad}(B))_{|\mathfrak{k}} \cdot \alpha)(X_\sigma) = (\mathrm{id}_{\mathfrak{k}} - \exp(\operatorname{ad}(B(\sigma))_{|\mathfrak{k}}) \cdot \alpha(X_\sigma)$$

with $\mathrm{id}_{\mathfrak{k}}$ and $\exp(\operatorname{ad}(B(\sigma)))$ as in Sec. 2.3 above.

Now observe that for $b \in \mathfrak{t}$ we have (cf. Eq. A.2 in Appendix A below)

$$\det(\mathrm{id}_{\mathfrak{k}} - \exp(\operatorname{ad}(b))_{|\mathfrak{k}}) = \prod_{\alpha \in \mathcal{R}_+} 4\sin^2(\pi\alpha(b)) \tag{2.20}$$

where $\mathcal{R}_+$ is the set of positive real roots of $(\mathfrak{g}, \mathfrak{t})$.

In view of Eqs. (2.11), (2.19), and (2.20) we now rewrite the informal determinant $\operatorname{Det}(B)$ in Eq. (2.18) as

$$\operatorname{Det}(B) = \prod_{\alpha \in \mathcal{R}_+} \det(O_\alpha^{(0)}(B))^2 \det(O_\alpha^{(1)}(B))^{-1}$$

where for each fixed $\alpha \in \mathcal{R}_+$ the operators $O_\alpha^{(i)}(B) : \Omega^i(\Sigma, \mathbb{R}) \to \Omega^i(\Sigma, \mathbb{R})$, $i = 0, 1$, are the multiplication operators obtained by multiplication with the function $\Sigma \ni \sigma \mapsto 2\sin(\pi\alpha(B(\sigma))) \in \mathbb{R}$.

Let us now equip the two spaces $\Omega^i(\Sigma, \mathbb{R})$, $i = 0, 1$, with the scalar product which is induced by the Riemannian metric $\mathbf{g}$ on Σ fixed in Sec. 2.3 above. By $\overline{\Omega^i(\Sigma, \mathbb{R})}$ we will denote the completion of the pre-Hilbert space $\Omega^i(\Sigma, \mathbb{R})$, $i = 0, 1$.

Let us now define

$$\operatorname{Det}_{rig}(B) := \prod_{\alpha \in \mathcal{R}_+} \operatorname{Det}_{rig,\alpha}(B) \tag{2.21a}$$

with

$$\operatorname{Det}_{rig,\alpha}(B) := \lim_{\epsilon \to 0} \left[\det{}_\epsilon(O_\alpha^{(0)}(B))^2 \det{}_\epsilon(O_\alpha^{(1)}(B))^{-1}\right] \tag{2.21b}$$

where for $i = 0, 1$ we have set[11]

$$\det{}_\epsilon(O_\alpha^{(i)}(B)) := \exp\left(\operatorname{Tr}\left(e^{-\epsilon\triangle_i} \log(O_\alpha^{(i)}(B))\right)\right) \tag{2.21c}$$

Here $\triangle_i$ is the Hodge Laplacian on $\overline{\Omega^i(\Sigma, \mathbb{R})}$ w.r.t. the Riemannian metric $\mathbf{g}$ on Σ and $\log : \mathbb{R}\backslash\{0\} \to \mathbb{C}$ is the restriction to $\mathbb{R}\backslash\{0\}$ of the principal branch of the complex logarithm. We remark that in the special case $B \in \mathcal{B}_{reg}$, which we will assume in the following, each of the bounded operators $O_\alpha^{(i)}(B)$, $i = 0, 1$, is a symmetric operator whose spectrum is bounded away from zero (cf. Eq. (A.1) in Appendix A) so $\log(O_\alpha^{(i)}(B))$ is a well-defined bounded operator. Moreover, since $e^{-\epsilon\triangle_i}$ is trace-class the product $e^{-\epsilon\triangle_i} \log(O_\alpha^{(i)}(B))$ is also trace-class and the expression $\operatorname{Tr}(e^{-\epsilon\triangle_i} \log(O_\alpha^{(i)}(B)))$ is therefore well-defined. Explicitly, we have

$$\operatorname{Tr}(e^{-\epsilon\triangle_i} \log(O_\alpha^{(i)}(B))) = \int_\Sigma \operatorname{Tr}(K_\epsilon^{(i)}(\sigma, \sigma)) \log(2\sin(\pi\alpha(B(\sigma)))) d\mu_{\mathbf{g}}(\sigma) \tag{2.22}$$

where $K_\epsilon^{(0)} : \Sigma \times \Sigma \to \mathbb{R} \cong \operatorname{End}(\mathbb{R})$ is the integral kernel of $e^{-\epsilon\triangle_0}$ and $K_\epsilon^{(1)} : \Sigma \times \Sigma \to \bigcup_{\sigma_1, \sigma_2 \in \Sigma} \operatorname{Hom}(T_{\sigma_1}\Sigma, T_{\sigma_2}\Sigma)$ is the integral kernel of $e^{-\epsilon\triangle_1}$. According to a famous result in [21] the negative powers of ϵ that appear in the asymptotic expansion of $K_\epsilon^{(i)}$, $i = 0, 1$ as $\epsilon \to 0$ cancel each other (= the "fantastic cancelations") and we obtain

$$\left[2\operatorname{Tr}(K_\epsilon^{(0)}(\sigma, \sigma)) - \operatorname{Tr}(K_\epsilon^{(1)}(\sigma, \sigma))\right] \to \tfrac{1}{4\pi} R_{\mathbf{g}}(\sigma) \quad \text{uniformly in } \sigma \text{ as } \epsilon \to 0 \tag{2.23}$$

[11]this ansatz is, of course, motivated by the rigorous formula $\det(A) = \exp(\operatorname{Tr}(\ln(A)))$ which holds for every strictly positive (self-adjoint) operator A on a finite-dimensional Hilbert-space

where $R_{\mathbf{g}}$ is the scalar curvature (= twice the Gaussian curvature) of $(\Sigma, \mathbf{g})$.

From Eqs. (2.21c), (2.22), and (2.23) it follows that the $\epsilon \to 0$ limit in Eq. (2.21b) really exists and that we have

$$\mathrm{Det}_{rig,\alpha}(B) = \exp\left(\int_{\Sigma} \log(2\sin(\pi\alpha(B(\sigma)))) \tfrac{1}{4\pi} R_{\mathbf{g}}(\sigma) d\mu_{\mathbf{g}}(\sigma) \right) \tag{2.24}$$

In the special case where $B \equiv b$ (with $b \in \mathfrak{t}_{reg}$) we can apply the classical Gauss-Bonnet Theorem

$$4\pi\chi(\Sigma) = \int_{\Sigma} R_{\mathbf{g}} d\mu_{\mathbf{g}} \tag{2.25}$$

where $\chi(\Sigma)$ is the Euler characteristic of Σ and obtain

$$\mathrm{Det}_{rig,\alpha}(B) = (2\sin(\pi\alpha(b)))^{\chi(\Sigma)} \tag{2.26}$$

and therefore

$$\mathrm{Det}_{rig}(B) = \big(\det\big(\mathrm{id}_{\mathfrak{k}} - \exp(\mathrm{ad}(b))_{|\mathfrak{k}}\big)\big)^{\chi(\Sigma)/2} \tag{2.27}$$

So in particular, the value of $\mathrm{Det}_{rig}(B)$ is independent of the auxiliary Riemannian metric $\mathbf{g}$ in this special case.

Remark 2.4 *i) In the special case where B is constant the calculation of $\mathrm{Det}_{rig}(B)$ just described can be simplified considerably. In particular, in this case Eq.* (2.23) *is not necessary for evaluating $\mathrm{Det}_{rig}(B)$ and proving Eq.* (2.27)*. Indeed, for constant B we only have to show that $\lim_{\epsilon \to 0}\big(2\,\mathrm{Tr}\big(e^{-\epsilon\triangle_0}\big) - \mathrm{Tr}\big(e^{-\epsilon\triangle_1}\big)\big) = \chi(\Sigma)$. But this follows because $\lim_{\epsilon \to 0}\big(2\,\mathrm{Tr}\big(e^{-\epsilon\triangle_0}\big) - \mathrm{Tr}\big(e^{-\epsilon\triangle_1}\big)\big) \overset{(*)}{=} 2\dim(\ker(\triangle_0)) - \dim(\ker(\triangle_1)) \overset{(**)}{=} 2\dim(H^0(\Sigma,\mathbb{R})) - \dim(H^1(\Sigma,\mathbb{R})) = \chi(\Sigma)$. Here step $(*)$ follows from another famous argument in [21] and step $(**)$ follows because according to the Hodge theorem we have $\ker(\triangle_i) \cong H^i(\Sigma,\mathbb{R})$.*

ii) We also mention that in the special case where B is constant there is an alternative way of defining and computing $\mathrm{Det}(B)$ using the Ray-Singer Torsion (which makes use of a suitable ζ-function regularization), cf. Sec. 3 in [2].

The special case $B \equiv b$ mentioned above was the only case which was relevant in [2] where only links consisting of "vertical" loops were studied (at a heuristic level). Here a "vertical" loop in $M = \Sigma \times S^1$ is a loop $l : S^1 \to \Sigma \times S^1$ which is "parallel" to S^1 (or in other words: $\mathrm{arc}(l_\Sigma)$ is just a point in Σ).

By contrast we will work with more general (ribbon) links which means that step functions B of the type

$$B = \sum\nolimits_{i=1}^{r} b_i 1_{Y_i}, \qquad r \in \mathbb{N} \tag{2.28}$$

will appear later during the explicit evaluation of $\mathrm{WLO}_{rig}(L)$ defined in Sec. 3.4 below, namely, after the $\epsilon \to 0$ and $s \to 0$-limits on the RHS of Eq. (3.21) below have been carried out. The regions $(Y_i)_{i \leq r}$ here are the r connected components of

$$\Sigma \backslash \mathrm{arc}(L_\Sigma) = \Sigma \backslash \bigcup\nolimits_{i=1}^{m} \mathrm{arc}(l^i_\Sigma) \tag{2.29}$$

In view of Eq. (2.27) above one would expect that for $B : \Sigma \to \mathfrak{t}$ of the form (2.28) one has[12]

$$\mathrm{Det}_{rig}(B) = \prod\nolimits_{i=1}^{r} \big(\det{}^{1/2}\big(\mathrm{id}_{\mathfrak{k}} - \exp(\mathrm{ad}(b_i))_{|\mathfrak{k}}\big)\big)^{\chi(Y_i)} \tag{2.30}$$

[12]observe that in contrast to $\chi(\Sigma)$, which is always an even number, $\chi(Y_i)$ can be odd, so in general we do *not* have $\big(\det^{1/2}\big(\mathrm{id}_{\mathfrak{k}} - \exp(\mathrm{ad}(b_i))_{|\mathfrak{k}}\big)\big)^{\chi(Y_i)} = \big(\det\big(\mathrm{id}_{\mathfrak{k}} - \exp(\mathrm{ad}(b_i))_{|\mathfrak{k}}\big)\big)^{\chi(Y_i)/2}$

where $\det^{1/2}(\mathrm{id}_{\mathfrak{k}} - \exp(\mathrm{ad}(\cdot))_{|\mathfrak{k}}) : \mathfrak{t} \to \mathbb{R}$ is given by

$$\det{}^{1/2}(\mathrm{id}_{\mathfrak{k}} - \exp(\mathrm{ad}(b))_{|\mathfrak{k}}) = \prod\nolimits_{\alpha \in \mathcal{R}+} 2\sin(\pi\alpha(b)) \tag{2.31}$$

And in fact, Eq. (2.30) is exactly the formula which is necessary for Conjecture 2 below to be true. The obvious question now is whether Eq. (2.30) follows from Eq. (2.21a) and Eq. (2.24) above[13].

In order to answer this question recall the following, more general version[14] of the classical Gauss-Bonnet Theorem mentioned above: Let $Y \subset \Sigma$ be such that the boundary ∂Y is (either empty or) a smooth 1-dimensional submanifold of Σ. We equip ∂Y with the Riemannian metric induced by $\mathbf{g} = \mathbf{g}_\Sigma$ and denote by ds the corresponding "line element" on ∂Y. Then we have

$$4\pi\chi(Y) = \int_Y R_{\mathbf{g}} d\mu_{\mathbf{g}} + 2\int_{\partial Y} k_{\mathbf{g}} ds \tag{2.32}$$

where $k_{\mathbf{g}}(p)$ for $p \in \partial Y$ is the geodesic curvature of ∂Y in the point p.

Let us now go back to the question whether Eq. (2.30) follows from Eq. (2.24). The short answer is: not for an arbitrary choice of $\mathbf{g}$ but for a natural subclass of the possible choices, cf. Assumption 2 and Remark 2.5 below. Observe that in Eq. (2.24) there is no term involving geodesic curvature. It is conceivable that such a term appears during the explicit evaluation of the RHS of Eq. (3.17) in Step 3 below as a result of "self linking". However, even in this case the "self linking" expressions which we obtain will depend on the precise regularization procedure which is used. We plan to study this issue in more detail in the near future (cf. Question 3 in Sec. 4 below). In the present paper we will bypass this issue by restricting ourselves to the situation where the auxiliary Riemannian metric $\mathbf{g}$ chosen above fulfills a suitable condition. One sufficient condition would be to assume that $\mathbf{g}$ is chosen such that the geodesic curvature of each of the sets $\mathrm{arc}(l^i_\Sigma)$ vanishes. In this case Eq. (2.30) follows indeed from Eq. (2.24). However, since at first look this condition may seem somewhat unnatural we will use the following assumption (which according to Remark 2.5 below is definitely natural) in the following:

Assumption 2 *From now on we will assume that the auxiliary Riemannian metric* $\mathbf{g}$ *on* Σ *was chosen such that on each* $\mathrm{Image}(\pi_\Sigma \circ R_i)$, $i \leq m$, *it coincides with the Riemannian metric "induced" by* $\pi_\Sigma \circ R_i : S^1 \times (0,1) \to \Sigma$. *Here we have equipped* $S^1 \times (0,1)$ *with the product of the standard (normalized) Riemannian metrics on* S^1 *and on* $(0,1)$.

We expect that Assumption 2 will lead to the correct values for the rigorous implementation $\mathrm{WLO}_{rig}(L)$ of $\mathrm{WLO}(L)$, which we will give below, cf. Conjectures 1 and 2 in Sec. 3.4 below. Moreover, the use of Assumption 2 eliminates the regularization dependence of the "self linking terms" we referred to above.

Remark 2.5 *Recall that the original heuristic path integral expression in Eq.* (2.5) *above is topologically invariant. In particular, it does not involve a Riemannian metric. However, for technical reasons, we later introduced an auxiliary Riemannian metric* $\mathbf{g}$ *breaking topological invariance.*

Clearly, in a situation where one introduces an auxiliary object $\mathcal{O}$ *in order to make sense of a heuristic expression one of the following two principles (or a combination of them) ought to be fulfilled in order to be able to claim to have a natural treatment:*

[13] observe that the RHS of Eq. (2.24) makes sense not only if $B : \Sigma \to \mathfrak{t}$ is smooth but also if B is measurable, bounded, and bounded away from $\mathfrak{t}_{sing} := \mathfrak{t} \backslash \mathfrak{t}_{reg}$, cf. Eq. (A.1) in Appendix A.

[14] In view of Remark 2.6 below we also remark that Eq. (2.32) can be generalized further to the situation where the boundary ∂Y is only a piecewise smooth (rather than a smooth) submanifold of Σ. In the generalized formula there will be an extra term on the RHS involving a sum over the finite number of points p of ∂Y where ∂Y is not smooth (and containing the corresponding "angle" of ∂Y at p).

i) The auxiliary object $\mathcal{O}$ can be chosen arbitrarily and the final result does not depend on it.

ii) There is a distinguished/canonical choice of $\mathcal{O}$ and that is the choice which we use.

Our Assumption 2 is a combination of these two principles. The restriction of $\mathbf{g}$ *on* $S := \bigcup_{i=1}^{m} \text{Image}(\pi_\Sigma \circ R_i)$ *is given canonically. On the other hand, the restriction of* $\mathbf{g}$ *on* $S^c := \Sigma \backslash S$ *can be essentially*[15] *chosen arbitrarily.*

Remark 2.6 If one wants to study the situation of general (ribbon) links L, i.e. links which need not fulfill Assumption 1 above, then Assumption 2 must be modified. This is because when $U := \text{Image}(\pi_\Sigma \circ R_i) \cap \text{Image}(\pi_\Sigma \circ R_j) \neq \emptyset$ for $i \neq j$ then in general each of the two maps $\pi_\Sigma \circ R_i$ and $\pi_\Sigma \circ R_j$ will induce a different Riemannian metric on U. One way to deal with this complication is to use, instead of Assumption 2, the aforementioned weaker condition[16] that $\mathbf{g}$ is chosen such that the geodesic curvature of each of the sets $\text{arc}(l^i_\Sigma)$ vanishes. Moreover, the generalization of Eq. (2.32) mentioned in Footnote 14 above will then be relevant.

2.6 The final heuristic formula (in the special case $\Sigma = S^2$)

Observe that

$$\mathcal{A}_c^\perp \cong \mathcal{A}_{\Sigma,\mathfrak{t}}$$

For simplicity we will assume in the following that $\Sigma \cong S^2$ and therefore $H^1(\Sigma) = \{0\}$. In this case the Hodge decomposition of $\mathcal{A}_{\Sigma,\mathfrak{t}}$ (w.r.t. the metric $\mathbf{g}$ fixed above) is given by

$$\mathcal{A}_{\Sigma,\mathfrak{t}} = \mathcal{A}_{ex} \oplus \mathcal{A}^*_{ex} \tag{2.33}$$

where

$$\mathcal{A}_{ex} := \{df \mid f \in C^\infty(\Sigma,\mathfrak{t})\} \tag{2.34}$$
$$\mathcal{A}^*_{ex} := \{\star df \mid f \in C^\infty(\Sigma,\mathfrak{t})\} \tag{2.35}$$

$\star$ being the relevant Hodge star operator. According to Eq. (2.33) we can replace the $\int \cdots DA_c^\perp$ integration in Eq. (2.17) by the integration $\int\int \cdots DA_{ex} DA^*_{ex}$ where DA_{ex}, DA^*_{ex} denote the heuristic "Lebesgue measures" on $\mathcal{A}_{ex}$ and $\mathcal{A}^*_{ex}$.

Taking this into account and replacing the heuristic expression $\text{Det}(B)$ in Eq. (2.18) by $\text{Det}_{rig}(B)$ as given by Eqs (2.21) we see that we can rewrite Eq. (2.17) as

$$\begin{aligned}\text{WLO}(L) \sim \lim_{s\to 0} \sum\nolimits_{y\in I} \int_{\mathcal{A}^*_{ex}\times C^\infty(\Sigma,\mathfrak{t})} \Big\{ \int_{\mathcal{A}_{ex}} I^{(s)}(L)(A_{ex} + A^*_{ex}, B) \Big\} DA_{ex} \Big\} \\ \times \exp(-2\pi i k \langle y, B(\sigma_0)\rangle) 1_{\mathcal{B}_{reg}}(B)\, \text{Det}_{rig}(B) \\ \times \exp(2\pi i k \ll \star A^*_{ex}, dB \gg_{\mathcal{A}^\perp})(DA^*_{ex} \otimes DB)\end{aligned} \tag{2.36}$$

where we have set

$$I^{(s)}(L)(A_c^\perp, B) := \int_{\check{\mathcal{A}}^\perp} \prod\nolimits_i \text{Tr}_{\rho_i}(\text{Hol}_{R_i^{(s)}}(\check{A}^\perp, A_c^\perp, B)))d\mu_B^\perp(\check{A}^\perp) \tag{2.37}$$

for $A_c^\perp \in \mathcal{A}_c^\perp \cong \mathcal{A}_{\Sigma,\mathfrak{t}}$ and $B \in \mathcal{B}$ and where we have used that

$$\ll \star A_c^\perp, d\check{B} \gg_{\mathcal{A}^\perp} = \ll \star A_c^\perp, d\check{B} \gg_{\mathcal{A}_\Sigma} = - \ll \star dA_c^\perp, \check{B} \gg_{L^2_\mathfrak{t}(\Sigma, d\mu_\mathbf{g})} \tag{2.38}$$

(which implies that $\ll \star A_{ex}, d\check{B} \gg_{\mathcal{A}^\perp} = 0$ if $A_{ex} \in \mathcal{A}_{ex}$).

[15] as long as $\mathbf{g}_{|S}$ and $\mathbf{g}_{|S^c}$ "fit together" smoothly, i.e. induce a smooth Riemannian metric on all of Σ

[16] this condition is arguably quite natural as well since it arises from Assumption 2 (which is natural according to Remark 2.5 above) by applying a suitable limit procedure

Using heuristic methods one can show[17]

$$I^{(s)}(L)(A_{ex} + A^*_{ex}, B) = I^{(s)}(L)(A^*_{ex}, B) \tag{2.39}$$

so, informally,

$$\int_{\mathcal{A}_{ex}} I^{(s)}(L)(A_{ex} + A^*_{ex}, B)\ DA_{ex} \sim I^{(s)}(L)(A^*_{ex}, B) \tag{2.40}$$

Moreover, if we introduce the decomposition $\mathcal{B} = \check{\mathcal{B}} \oplus \mathcal{B}_c$ where

$$\check{\mathcal{B}} := \{B \in \mathcal{B} \mid \int_\Sigma B\ d\mu_{\mathbf{g}} = 0\} \tag{2.41}$$

$$\mathcal{B}_c := \{B \in \mathcal{B} \mid B \text{ is constant}\} \cong \mathfrak{t} \tag{2.42}$$

we can also replace $\int \cdots DB$ by $\int\int \cdots D\check{B}db$ where $D\check{B}$ is the heuristic "Lebesgue measure" on $\check{\mathcal{B}}$ and db is the (rigorous) normalized Lebesgue measure on $\mathcal{B}_c \cong \mathfrak{t}$. Taking this into account we obtain from Eqs. (2.36) and (2.40)

$$\begin{aligned}\mathrm{WLO}(L) \sim \lim_{s\to 0} \sum\nolimits_{y\in I} \Big[\int_{\mathcal{B}_c} db \Big[\int_{\mathcal{A}^*_{ex}\times\check{\mathcal{B}}} & I^{(s)}(L)(A^*_{ex}, \check{B}+b)\exp(-2\pi i k\langle y, \check{B}(\sigma_0)+b\rangle) \\ & 1_{\mathcal{B}_{reg}}(\check{B}+b)\,\mathrm{Det}_{rig}(\check{B}+b) \\ & \times \exp(-2\pi i k \ll \star A^*_{ex}, d\check{B} \gg_{\mathcal{A}^\perp})(DA^*_{ex}\otimes D\check{B})\Big]\Big]\end{aligned} \tag{2.43}$$

Observe that the operator $\star d : \check{\mathcal{B}} \to \mathcal{A}^*_{ex}$ is a linear isomorphism. We can therefore make the change of variable $\check{B}_1 := (\star d)^{-1}A^*_{ex}$ and $\check{B}_2 := \check{B}$ and rewrite Eq. (2.43) as

$$\mathrm{WLO}(L) \sim \lim_{s\to 0} \sum\nolimits_{y\in I} \Big[\int_{\mathcal{B}_c} \Big[\int_{\check{\mathcal{B}}\times\check{\mathcal{B}}} J^{(s)}_{b,y}(L)(\check{B}_1, \check{B}_2)\ d\nu(\check{B}_1, \check{B}_2)\Big] db\Big] \tag{2.44}$$

where we have set

$$J^{(s)}_{b,y}(L)(\check{B}_1, \check{B}_2) := I^{(s)}(L)(\star d\check{B}_1, \check{B}_2 + b)\exp(-2\pi i k\langle y, \check{B}_2(\sigma_0)+b\rangle)1_{\mathcal{B}_{reg}}(\check{B}_2+b)\,\mathrm{Det}_{rig}(\check{B}_2+b) \tag{2.45}$$

and (cf. Eq. (2.38) above)

$$d\nu(\check{B}_1, \check{B}_2) := \tfrac{1}{Z}\exp(-2\pi i k \ll \star d \star d\check{B}_1, \check{B}_2 \gg_{L^2_{\mathfrak{t}}(\Sigma, d\mu_{\mathbf{g}})})(D\check{B}_1 \otimes D\check{B}_2) \tag{2.46}$$

with[18]

$$Z := \int \exp(-2\pi i k \ll \star d \star d\check{B}_1, \check{B}_2 \gg_{L^2_{\mathfrak{t}}(\Sigma, d\mu_{\mathbf{g}})})(D\check{B}_1 \otimes D\check{B}_2)$$

3 Rigorous realization of the RHS of Eq. (2.44)

We will now explain how one can make rigorous sense of the path integral expression appearing on the RHS of Eq. (2.44) within the framework of White Noise Analysis. In order to do so we will proceed in four steps:

Step 1: We make rigorous sense of the integral functional $\int \cdots d\mu^\perp_B$ appearing in Eq. (2.37).

[17] this is easy if L fulfills Assumption 1, cf. also Remark 3.1 below for a rigorous argument. For general (ribbon) links L this point is not yet clear, cf. Sec. 4 below

[18] Clearly, we could drop the factor $1/Z$ in Eq. (2.46) since the symbol $\sim$ in Eq. (2.44) denotes equality up to a multiplicative constant. In Sec. 3.2 we introduce a rigorous version of the integral functional $\int \cdots d\nu$ and there it is convenient that $d\nu$ is normalized.

Step 2: We make rigorous sense of the integral functional $\int \cdots \, d\nu$ appearing in Eq. (2.44).

Step 3: We make rigorous sense of the integral expression appearing in Eq. (2.37) above, i.e. of

$$I^{(s)}(L)(A_c^{\perp}, B) = \int_{\check{\mathcal{A}}^{\perp}} \prod_i \mathrm{Tr}_{\rho_i}(\mathrm{Hol}_{R_i^{(s)}}(\cdot, A_c^{\perp}, B))) d\mu_B^{\perp} \tag{3.1}$$

for $A_c^{\perp} \in \mathcal{A}_c^{\perp}$ and $B \in \mathcal{B}$.

Step 4: We make rigorous sense of the total expression on the RHS of Eq. (2.44).

3.1 Step 1

We will now give a rigorous implementation of the integral functional $\int \cdots d\mu_B^{\perp}$ appearing in Eq. (2.37) above as a generalized distribution $\Phi_B^{\perp}$ on a suitable extension $\overline{\check{\mathcal{A}}^{\perp}}$ of $\check{\mathcal{A}}^{\perp}$.

i) First we will choose a suitable Gelfand triple $(\mathcal{N}, \mathcal{H}_{\mathcal{N}}, \mathcal{N}')$ and set $\overline{\check{\mathcal{A}}^{\perp}} := \mathcal{N}'$.

Before we do this recall that

$$\mathcal{A}^{\perp} \cong C^{\infty}(S^1, \mathcal{A}_{\Sigma})$$
$$\check{\mathcal{A}}^{\perp} = \{A^{\perp} \in \mathcal{A}^{\perp} \mid \int A^{\perp}(t) dt \in \mathcal{A}_{\Sigma,\mathfrak{k}}\}$$

We set $\mathcal{H}_{\Sigma} := L^2\text{-}\Gamma(\mathrm{Hom}(T\Sigma, \mathfrak{g}), d\mu_{\mathbf{g}})$, i.e. $\mathcal{H}_{\Sigma}$ is the Hilbert space of L^2-sections (w.r.t. the measure $d\mu_{\mathbf{g}}$) of the bundle $\mathrm{Hom}(T\Sigma, \mathfrak{g}) \cong T^*\Sigma \otimes \mathfrak{g}$ equipped with the fiber metric induced by $\mathbf{g}$ and $\langle \cdot, \cdot \rangle$. Moreover, we set[19]

$$\mathcal{H}^{\perp} := L^2_{\mathcal{H}_{\Sigma}}(S^1, dt)$$
$$\check{\mathcal{H}}^{\perp} := \{H^{\perp} \in \mathcal{H}^{\perp} \mid \int H^{\perp}(t) dt \in \mathcal{H}_{\Sigma,\mathfrak{k}}\}$$

where $\mathcal{H}_{\Sigma,\mathfrak{k}}$ is the Hilbert space defined in a completely analogous way as $\mathcal{H}_{\Sigma}$ but with $\mathfrak{k}$ playing the role of $\mathfrak{g}$.

The Gelfand triple $(\mathcal{N}, \mathcal{H}_{\mathcal{N}}, \mathcal{N}')$ we choose is given by

$$\mathcal{N} := \check{\mathcal{A}}^{\perp} \tag{3.2}$$
$$\mathcal{H}_{\mathcal{N}} := \check{\mathcal{H}}^{\perp} \tag{3.3}$$

where we have equipped $\check{\mathcal{A}}^{\perp}$ with a suitable[20] family of semi-norms.

Using "second quantization" and the Wiener-Ito-Segal isomorphism

$$Fock_{sym}(\mathcal{H}_{\mathcal{N}}) \cong L^2_{\mathbb{C}}(\mathcal{N}', \gamma_{\mathcal{N}'})$$

where $\gamma_{\mathcal{N}'}$ is the canonical Gaussian measure on $\mathcal{N}'$ (associated to $(\mathcal{N}, \mathcal{H}_{\mathcal{N}}, \mathcal{N}')$) we obtain a new Gelfand triple $((\mathcal{N}), L^2_{\mathbb{C}}(\mathcal{N}', \gamma_{\mathcal{N}'}), (\mathcal{N})')$.

ii) Next we evaluate the Fourier transform $\mathcal{F}\mu_B^{\perp}$ of the informal measure $\mu_B^{\perp}$. From Eq. (2.13) and Eq. (2.16) we obtain immediately

$$\forall j \in \mathcal{N}: \quad \mathcal{F}\mu_B^{\perp}(j) = \int \exp(i \ll \cdot, j \gg_{\mathcal{H}_{\mathcal{N}}}) d\mu_B^{\perp} = \exp(-\tfrac{1}{2} \ll j, C_B j \gg_{\mathcal{H}_{\mathcal{N}}})$$

[19] In other words: $\mathcal{H}^{\perp}$ is the space of $\mathcal{H}_{\Sigma}$-valued (measurable) functions on S^1 which are square-integrable w.r.t. dt.

[20] more precisely, the family of semi-norms must be chosen such that $\mathcal{N} = \check{\mathcal{A}}^{\perp}$ is nuclear and the inclusion map $\mathcal{N} \to \mathcal{H}_{\mathcal{N}}$ is continuous

where C_B is given informally by

$$C_B = \big(-2\pi k \star (\partial/\partial t + \mathrm{ad}(B))\big)^{-1} = \tfrac{1}{2\pi k} \star (\partial/\partial t + \mathrm{ad}(B))^{-1} \tag{3.4}$$

For each fixed $B \in \mathcal{B}_{reg}$ we will now make sense of $(\partial/\partial t + \mathrm{ad}(B))^{-1}$ as a densely defined linear operator on $\check{\mathcal{H}}^{\perp}$. In order to do so we first introduce the space $\check{C}^{\infty}(S^1, \mathfrak{g}) := \{f \in C^{\infty}(S^1, \mathfrak{g}) | \int f(t)dt \in \mathfrak{k}\}$. It is not difficult to see that for $b \in \mathfrak{t}_{reg}$ the operator $\partial/\partial t + \mathrm{ad}(b) : \check{C}^{\infty}(S^1, \mathfrak{g}) \to \check{C}^{\infty}(S^1, \mathfrak{g})$ is invertible[21].

Now let $(\partial/\partial t + \mathrm{ad}(B))^{-1} : \check{\mathcal{A}} \to \check{\mathcal{A}} \subset \check{\mathcal{H}}^{\perp}$ be the linear operator given by

$$\big((\partial/\partial t + \mathrm{ad}(B))^{-1} \cdot \check{A}^{\perp})(t)\big)(X_{\sigma}) = (\partial/\partial t + \mathrm{ad}(B(\sigma))^{-1} \cdot \big(\check{A}^{\perp}(t)(X_{\sigma})\big) \tag{3.6}$$

for all $\check{A}^{\perp} \in \check{\mathcal{A}}^{\perp}$, $t \in S^1$, $\sigma \in \Sigma$, and $X_{\sigma} \in T_{\sigma}\Sigma$. Observe that $(\partial/\partial t + \mathrm{ad}(B))^{-1} \cdot \check{A}^{\perp}$ is indeed a well-defined element of $\check{\mathcal{A}}^{\perp}$ because, by the assumption on B we have $B \in \mathcal{B}_{reg}$, i.e. $B(\sigma) \in \mathfrak{t}_{reg}$ for all $\sigma \in \Sigma$.

It is easy to check that $(\partial/\partial t + \mathrm{ad}(B))^{-1}$ is bounded and anti-symmetric. Since $\star$ is bounded and anti-symmetric as well we have now found a rigorous realization of the operator C_B in Eq. (3.4) as a bounded and symmetric operator on $\check{\mathcal{H}}^{\perp}$.

For technical reasons we need to define[22] C_B also for $B \notin \mathcal{B}_{reg}$. In view of the indicator function $1_{\mathcal{B}_{reg}}$ appearing in Eq. (3.21) it seems[23] that we are entitled to define C_B in an arbitrary way if $B \notin \mathcal{B}_{reg}$. For simplicity we will take C_B to be trivial (i.e. $C_B = 0$) if $B \notin \mathcal{B}_{reg}$.

iii) For fixed $B \in \mathcal{B}$ let $U_B : \mathcal{N} \to \mathbb{C}$ be the well-defined continuous function given by

$$U_B(j) = \exp(-\tfrac{1}{2} \ll j, C_B j \gg_{\mathcal{H}_{\mathcal{N}}}) \tag{3.7}$$

for every $j \in \mathcal{N}$. It is straightforward to show that the function $U_B : \mathcal{N} \to \mathbb{C}$ is a "U-functional" in the sense of [16, 17]. In view of the Kondratiev-Potthoff-Streit Characterization Theorem (cf. again [16, 17]) the integral functional $\Phi_B^{\perp} := \int \cdots d\mu_B^{\perp}$ can be defined rigorously as the unique element $\Phi_B^{\perp}$ of $(\mathcal{N})'$ such that

$$\Phi_B^{\perp}(\exp(i(\cdot, j)_{\mathcal{N}})) = U_B(j) \tag{3.8}$$

holds for all $j \in \mathcal{N}$. Here $(\cdot, \cdot)_{\mathcal{N}} : \mathcal{N}' \times \mathcal{N} \to \mathbb{R}$ is the canonical pairing.

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[21] Its inverse $(\partial/\partial t + \mathrm{ad}(b))^{-1}$ is given explicitly by (cf. Eq. (5.8) in [12])

$$((\partial/\partial t + \mathrm{ad}(b))^{-1} f)(t) = T(b) \cdot \int_0^1 e^{s\,\mathrm{ad}(b)} f(t + i_{S^1}(s)) ds \tag{3.5}$$

for all $f \in \check{C}^{\infty}(S^1, \mathfrak{g})$ and $t \in S^1$ where $i_{S^1} : [0,1] \ni s \mapsto e^{2\pi i s} \in U(1) \cong S^1$ and where $T(b) \in \mathrm{End}(\mathfrak{g})$ is given by $T(b)(X) = (e^{\mathrm{ad}(b)} - \pi_{\mathfrak{k}})^{-1}(X)$ if $X \in \mathfrak{k}$ and $T(b)(X) = X$ if $X \in \mathfrak{t}$. Here $\pi_{\mathfrak{k}} : \mathfrak{g} \to \mathfrak{k}$ is the orthogonal projection. We remark that in the special case where f takes only values in $\mathfrak{t}$ Eq. (3.5) reduces to $((\partial/\partial t + \mathrm{ad}(b))^{-1} f)(t) = ((\partial/\partial t)^{-1} f)(t) = \int_0^1 f(t + i_{S^1}(s)) ds$

[22] observe that in Sec. 3.4 below we replace the indicator function $1_{\mathcal{B}_{reg}}$ by regularized versions $1^{(n)}_{\mathcal{B}_{reg}}$ and the condition $1^{(n)}_{\mathcal{B}_{reg}}(B) \neq 0$ (where $n \in \mathbb{N}$ is fixed) does no longer guarantee that $B \in \mathcal{B}_{reg}$

[23] In fact, in spite of this indicator function $1_{\mathcal{B}_{reg}}$ some of the functions B which appear during the explicit evaluation of the RHS of Eq. (3.21) will not be elements of $\mathcal{B}_{reg}$ (for reasons explained in the previous footnote). It would be more satisfactory to modify our approach in a suitable way, for example by using the regularization procedure described in Appendix B already now (and not only in Step 4), and working with $\Psi^{\perp}_{B^{(n)}}$ instead of $\Psi^{\perp}_B$ with $B^{(n)}$ given by Eq. (B.2) below. In the situation of ribbon links L fulfilling Assumption 1 above we can bypass this problem by simply defining C_B by $C_B := 0$ if $B \in \mathcal{B}_{reg}$. With this choice Eq. (3.18) in Proposition 3.2 will hold for all $B \in \mathcal{B}$ and we can expect Conjectures 1 and 2 to be true.

Convention 2 *Let $\pi : \mathcal{A}^\perp = \check{\mathcal{A}}^\perp \oplus \mathcal{A}_c^\perp \to \check{\mathcal{A}}^\perp$ be the canonical projection. The map $\pi' : (\check{\mathcal{A}}^\perp)' \to (\mathcal{A}^\perp)'$ which is dual to π is an injection. Using π' we will identify $(\check{\mathcal{A}}^\perp)'$ with a subspace of $(\mathcal{A}^\perp)'$.*

Moreover, we will identify each element $A^\perp$ of $(\mathcal{A}^\perp)'$ with the continuous map $f_{A^\perp} : \mathcal{A}_{\mathbb{R}}^\perp \to \mathfrak{g}$ given by $f_{A^\perp}(\psi^\perp) = \sum_a T_a(A^\perp, T_a\psi^\perp)$ for all $\psi^\perp \in \mathcal{A}_{\mathbb{R}}^\perp$ where $\mathcal{A}_{\mathbb{R}}^\perp := C^\infty(S^1, \mathcal{A}_{\Sigma,\mathbb{R}})$ and where $(T_a)_a$ is any fixed $\langle\cdot,\cdot\rangle$-orthonormal basis of $\mathfrak{g}$.

3.2 Step 2

In order to make rigorous sense of the heuristic integral functional $\int_{\check{\mathcal{B}}\times\check{\mathcal{B}}} \cdots d\nu$ as a generalized distribution on a suitable extension $\overline{\check{\mathcal{B}} \times \check{\mathcal{B}}}$ of the space $\check{\mathcal{B}} \times \check{\mathcal{B}}$ we will proceed in a similar way as in Step 1 above.

i) First we choose a suitable Gelfand triple $(\mathcal{E}, \mathcal{H}_{\mathcal{E}}, \mathcal{E}')$ and set

$$\overline{\check{\mathcal{B}} \times \check{\mathcal{B}}} := \mathcal{E}'.$$

More precisely, we choose

$$\mathcal{E} := \check{\mathcal{B}} \times \check{\mathcal{B}} \tag{3.9}$$

$$\mathcal{H}_{\mathcal{E}} := \check{L}_{\mathfrak{t}}^2(\Sigma, d\mu_{\mathbf{g}}) \oplus \check{L}_{\mathfrak{t}}^2(\Sigma, d\mu_{\mathbf{g}}) \tag{3.10}$$

where we have equipped $\mathcal{E}$ with a suitable family of semi-norms and where we have set $\check{L}_{\mathfrak{t}}^2(\Sigma, d\mu_{\mathbf{g}}) := \{f \in L_{\mathfrak{t}}^2(\Sigma, d\mu_{\mathbf{g}}) \mid \int f d\mu_{\mathbf{g}} = 0\}$. Using second quantization and the Wiener-Ito-Segal isomorphism

$$Fock_{sym}(\mathcal{H}_{\mathcal{E}}) \cong L_{\mathbb{C}}^2(\mathcal{E}', \gamma_{\mathcal{E}'})$$

where $\gamma_{\mathcal{E}'}$ is the canonical Gaussian measure on $\mathcal{E}'$ (associated to $(\mathcal{E}, \mathcal{H}_{\mathcal{E}}, \mathcal{E}')$) we obtain a new Gelfand triple $((\mathcal{E}), L_{\mathbb{C}}^2(\mathcal{E}', \gamma_{\mathcal{E}'}), (\mathcal{E})')$.

ii) Next we evaluate the Fourier transform $\mathcal{F}\nu$ of the heuristic "measure" ν at an informal level. Clearly, ν is of "Gauss type" with the well-defined[24] covariance operator

$$C := -\tfrac{1}{2\pi k}\begin{pmatrix} 0 & (\triangle_{|\check{\mathcal{B}}})^{-1} \\ (\triangle_{|\check{\mathcal{B}}})^{-1} & 0 \end{pmatrix}, \qquad \text{where } \triangle := \star d \star d$$

Taking this into account we obtain

$$\forall j \in \mathcal{E}: \quad \mathcal{F}\nu(j) = \int \exp(i \ll \cdot, j \gg_{\mathcal{H}_{\mathcal{E}}}) d\nu = \exp(-\tfrac{1}{2} \ll j, Cj \gg_{\mathcal{H}_{\mathcal{E}}})$$

We remark that C is a (densely defined) bounded and symmetric linear operator on $\mathcal{H}_{\mathcal{E}}$.

iii) Let $U : \mathcal{E} \to \mathbb{C}$ be given by

$$U(j) = \exp(-\tfrac{1}{2} \ll j, Cj \gg_{\mathcal{H}_{\mathcal{E}}}) \tag{3.11}$$

for every $j \in \mathcal{E}$. Clearly, $U : \mathcal{E} \to \mathbb{C}$ is a "U-functional" in the sense of [16, 17] so using the Kondratiev-Potthoff-Streit Characterization Theorem the integral functional $\Psi := \int_{\check{\mathcal{B}}\times\check{\mathcal{B}}} \cdots d\nu$ can be defined rigorously as the unique element Ψ of $(\mathcal{E})'$ such that

$$\Psi(\exp(i(\cdot, j)_{\mathcal{E}})) = U(j) \tag{3.12}$$

holds for all $j \in \mathcal{E}$. Here $(\cdot,\cdot)_{\mathcal{E}} : \mathcal{E}' \times \mathcal{E} \to \mathbb{R}$ is the canonical pairing.

Convention 3 *Let $\pi : \mathcal{B} = \check{\mathcal{B}} \oplus \mathcal{B}_c \to \check{\mathcal{B}}$ be the canonical projection. The map $\pi' : \check{\mathcal{B}}' \to \mathcal{B}'$ which is dual to π is an injection. Using π' we will identify $\check{\mathcal{B}}'$ with a subspace of $\mathcal{B}'$.*

Moreover, we will identify each element B of $\mathcal{B}'$ with the continuous map $f_B : C^\infty(\Sigma, \mathbb{R}) \to \mathfrak{t}$ given by $f_B(\psi) = \sum_a T_a(B, T_a\psi)_{\mathcal{B}}$ for all $\psi \in C^\infty(\Sigma, \mathbb{R})$ where $(\cdot,\cdot)$ is the canonical pairing $\mathcal{B} \times \mathcal{B}' \to \mathbb{R}$ and $(T_a)_a$ is a fixed orthonormal basis of $\mathfrak{t}$.

[24] observe that the kernel of $\triangle : \mathcal{B} \to \mathcal{B}$ equals $\mathcal{B}_c$, so $\triangle_{|\check{\mathcal{B}}}$ is injective

3.3 Step 3

Let us now make rigorous sense of the heuristic integral

$$I^{(s)}(L)(A_c^{\perp}, B) = \int_{\check{\mathcal{A}}^{\perp}} \prod_i \mathrm{Tr}_{\rho_i}(\mathrm{Hol}_{R_i^{(s)}}(\check{A}^{\perp}, A_c^{\perp}, B))) d\mu_B^{\perp}(\check{A}^{\perp}) \tag{3.13}$$

for $A_c^{\perp} \in \mathcal{A}_c^{\perp}$ and $B \in \mathcal{B}$. We already have a rigorous version $\Phi_B^{\perp}$ of the heuristic integral functional $\int \cdots d\mu_B^{\perp}$. However, clearly we can not just consider

$$\Phi_B^{\perp}\big(\prod_i \mathrm{Tr}_{\rho_i}\big(\mathrm{Hol}_{R_i^{(s)}}(\cdot, A_c^{\perp}, B)\big)\big)$$

since the function $\mathrm{Hol}_{R_i^{(s)}}(\cdot, A_c^{\perp}, B)$ was defined as a function on $\check{\mathcal{A}}^{\perp}$ and not as a function on all of $\overline{\check{\mathcal{A}}^{\perp}} = \mathcal{N}'$.

Let us now fix $j \leq m$ and $s > 0$ temporarily (until Proposition 3.1) and set $R := R_j^{(s)}$. Using[25] $\check{A}^{\perp}((R_u)'(t)) = \check{A}^{\perp}((\pi_\Sigma \circ R_u)'(t))$ we obtain

$$\begin{aligned}
&\mathrm{Hol}_R(\check{A}^{\perp}, A_c^{\perp}, B) \\
&= \lim_{n\to\infty} \prod_{j=1}^{n} \exp\big(\tfrac{1}{n} \int_0^1 \big[(\check{A}^{\perp} + A_c^{\perp} + Bdt)(R_u'(t))\big] du\big)_{|t=j/n} \\
&= \lim_{n\to\infty} \prod_{j=1}^{n} \exp\big(\tfrac{1}{n} \int_0^1 \big[\check{A}^{\perp}((\pi_\Sigma \circ R_u)'(t)) + (A_c^{\perp} + Bdt)(R_u'(t))\big] du\big)_{|t=j/n}
\end{aligned} \tag{3.14}$$

Clearly, for a general element $\check{A}^{\perp}$ of $\overline{\check{\mathcal{A}}^{\perp}} = \mathcal{N}'$ the expression $\check{A}^{\perp}((\pi_\Sigma \circ R_u)'(t))$ appearing in the last expression does not make sense.

In order to get round this complication we will now make use of "point smearing", i.e. replace points by suitable test functions.

In order to do so we choose, for each $\sigma \in \Sigma$ a Dirac family[26] $(\delta_\sigma^\epsilon)_{\epsilon>0}$ around σ w.r.t. $d\mu_{\mathbf{g}}$. Moreover, for each $t \in S^1$ we choose a Dirac family $(\delta_t^\epsilon)_{\epsilon>0}$ around t w.r.t. the measure dt on S^1.

For every $p = (\sigma, t) \in \Sigma \times S^1$ and $\epsilon > 0$ we define $\delta_p^\epsilon \in C^\infty(\Sigma \times S^1, \mathbb{R})$ by

$$\delta_p^\epsilon(\sigma', t') := \delta_\sigma^\epsilon(\sigma')\delta_t^\epsilon(t') \quad \text{ for all } \sigma' \in \Sigma \text{ and } t' \in S^1.$$

For technical reasons we will assume[27] also that for each fixed $\epsilon > 0$ the family $(\delta_\sigma^\epsilon)_{\sigma\in\Sigma}$ was chosen such that the function $\Sigma \times \Sigma \ni (\sigma, \bar{\sigma}) \to \delta_\sigma^\epsilon(\bar{\sigma}) \in \mathbb{R}$ is smooth and, moreover, to have the property that for each ϵ and $\sigma \in \Sigma$ the support of δ_σ^ϵ is contained in the ϵ-ball w.r.t. $d_{\mathbf{g}}$ around σ.

Recall that $R = R_j^{(s)}$, $j \leq m$, $s > 0$. Let us now also introduce the notation $\bar{R} := R_j$. Let $\epsilon_j(s)$ be the supremum of all $\epsilon > 0$ such that for all $t \in S^1$, and $u \in [0,1]$ we have[28] $\mathrm{supp}(\delta_{R_u(t)}^\epsilon) \subset \bar{R}_\Sigma$. Let $X_{\bar{R}_\Sigma}$ be the vector field on $\mathrm{Image}(\bar{R}_\Sigma) \subset \Sigma$, which is induced by the collection of loops $S^1 \ni t \mapsto \bar{R}_\Sigma(t, u) \in \Sigma$, $u \in [0,1]$.

After these preparations we can now introduce "smeared" analogues for the expression $\check{A}^{\perp}((\pi_\Sigma \circ R_u)'(t))$ appearing in Eq. (3.14) above. More precisely, we now replace, for fixed $\epsilon \in (0, \epsilon_j(s))$ the expression $\check{A}^{\perp}((\pi_\Sigma \circ R_u)'(t))$ by the expression $\check{A}^{\perp}(X_{\bar{R}_\Sigma}\delta_{R_u(t)}^\epsilon)$. Here we

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[25] of course, we also have $A_c^{\perp}((R_u)'(t)) = A_c^{\perp}((\pi_\Sigma \circ R_u)'(t))$ but we will not need this

[26] i.e. for each fixed $\sigma \in \Sigma$ we have the following: δ_σ^ϵ, $\epsilon > 0$, is a non-negative and smooth function $\Sigma \to \mathbb{R}$. Moreover, $\int \delta_\sigma^\epsilon d\mu_{\mathbf{g}} = 1$, and we have $\delta_\sigma^\epsilon \to \delta_\sigma$ weakly as $\epsilon \to 0$ where δ_σ is the Dirac distribution in the point σ

[27] in view of Question 2 in Sec. 4 below we remark that if we want to treat the case of general ribbon links L (i.e. ribbon links for which Assumption 1 need not be fulfilled) we will probably have to make some additional technical assumptions on the family $(\delta_\sigma^\epsilon)_{\sigma\in\Sigma}$

[28] observe that $\delta_{R_u(t)}^\epsilon$ depends on s even though this is not reflected in the notation

made the identification $VF(\Sigma) \cong \mathcal{A}_{\Sigma,\mathbb{R}}$ using the Riemannian metric $\mathbf{g}$. On the other hand $\mathcal{A}_{\Sigma,\mathbb{R}} \subset C^\infty(S^1, \mathcal{A}_{\Sigma,\mathbb{R}}) \cong \mathcal{A}^\perp_{\mathbb{R}}$, so according to Convention 2 above the expression $\check{A}^\perp(X_{\bar{R}_\Sigma} \delta^\epsilon_{R_u(t)})$ is well-defined for every $\check{A}^\perp \in \overline{\check{\mathcal{A}}^\perp} = \mathcal{N}'$. We can now set

$$\mathrm{Hol}^{(\epsilon)}_R(\check{A}^\perp, A^\perp_c, B) :=$$

$$\lim_{n\to\infty} \prod_{j=1}^n \exp\Big(\tfrac{1}{n} \int_0^1 \big[\check{A}^\perp(X_{\bar{R}_\Sigma} \delta^\epsilon_{R_u(t)}) + (A^\perp_c + Bdt)(R'_u(t))\big] du\Big)_{|t=j/n} \in G \quad (3.15)$$

Using similar methods as in the proof of Proposition 6 in [7] it is not difficult to prove the following result:

Proposition 3.1 *For every $s \in (0,1)$ and every $\epsilon \in (0, \epsilon(s))$ where $\epsilon(s) := \min_{i\leq m} \epsilon_i(s)$ we have*

$$\prod_i \mathrm{Tr}_{\rho_i}\big(\mathrm{Hol}^{(\epsilon)}_{R_i^{(s)}}(\cdot, A^\perp_c, B)\big) \in (\mathcal{N}) \quad (3.16)$$

Consequently, the expression

$$I^{(s,\epsilon)}_{rig}(L)(A^\perp_c, B) := \Phi^\perp_B\Big(\prod_i \mathrm{Tr}_{\rho_i}\big(\mathrm{Hol}^{(\epsilon)}_{R_i^{(s)}}(\cdot, A^\perp_c, B)\big)\Big) \quad (3.17)$$

is well-defined.

Proposition 3.2 *For every L fulfilling Assumption 1 above we have*[29]

$$I^{(s,\epsilon)}_{rig}(L)(A^\perp_c, B) = \prod_i \mathrm{Tr}_{\rho_i}\left(\exp\left(\int_0^1 \left(\int_{(R_i^{(s)})_u} (A^\perp_c + Bdt)\right) du\right)\right) \quad (3.18)$$

where for a loop $l : S^1 \to M$ and a 1-form α on M we use the notation $\int_l \alpha = \int_{S^1} l^*(\alpha)$. Observe that the RHS of Eq. (3.18) actually does not depend on ϵ.

Remark 3.1 *We remark that Eq. (3.18) implies that $I^{(s,\epsilon)}_{rig}(L)(A_{ex}+A^*_{ex}, B) = I^{(s,\epsilon)}_{rig}(L)(A^*_{ex}, B)$ (cf. the notation in Sec. 2.6 above) which can be seen as a rigorous justification of Eq. (2.39) in Sec. 2.6 above.*

Remark 3.2 i) Here is an alternative way for defining a "smeared" analogue of the expression $\check{A}^\perp((\pi_\Sigma \circ R_u)'(t))$. Observe that since Σ is compact there is a $\epsilon_0 > 0$ such that for all $\sigma_0, \sigma_1 \in \Sigma$ with $d_{\mathbf{g}}(\sigma_0, \sigma_1) < \epsilon_0$ there is a unique (geodesic) segment starting in σ_0 and ending in σ_1. Using parallel transport along this geodesic segment w.r.t. the Levi-Civita connection of $(\Sigma, \mathbf{g})$ we can transport every tangent vector $v \in T_{\sigma_0}\Sigma$ to a tangent vector in $T_{\sigma_1}\Sigma$. Thus every $v \in T_{\sigma_0}\Sigma$ induces in a natural way a vector field X_v on the open ball $B_{\epsilon_0}(\sigma_0) \subset \Sigma$.

For every $\epsilon < \epsilon_0$ we replace $\check{A}^\perp((\pi_\Sigma \circ R_u)'(t))$ by the expression $\check{A}^\perp(X_{(\pi_\Sigma\circ R_u)'(t)} \delta^\epsilon_{R_u(t)})$.

ii) The alternative method also works when L does not fulfill Assumption 1 while the original method must be modified (in a relatively straightforward way) if L does not fulfill Assumption 1.

iii) When both Assumption 1 and Assumption 2 are fulfilled then both methods described here are equivalent. Indeed, for each fixed t and u the vector field $X_{(\pi_\Sigma\circ R_u)'(t)}$ coincides with the vector field $X_{\bar{R}_\Sigma}$ on the subset $S \subset \Sigma$ where both vector fields are defined. For sufficiently small $\epsilon > 0$ we therefore have $X_{(\pi_\Sigma\circ R_u)'(t)} \delta^\epsilon_{R_u(t)} = X_{\bar{R}_\Sigma} \delta^\epsilon_{R_u(t)}$.

iv) If Assumption 2 is not fulfilled then the two methods described here will probably not be equivalent. In particular, it seems that non-trivial self-linking terms appear when using the alternative method while no such self-linking term will arise when the original method is used.

[29]recall that we also assume that Assumption 2 is fulfilled

3.4 Step 4

Finally, let us make rigorous sense of the full heuristic expressions on the RHS of Eq. (2.44) above.

For similar reasons as in Step 3 above we will use again "point smearing". Recall that above we chose for each $\sigma \in \Sigma$ a "Dirac family" $(\delta^{\epsilon}_{\sigma})_{\epsilon>0}$ such that for every $\epsilon > 0$ the function $\Sigma \times \Sigma \ni (\sigma, \bar{\sigma}) \to \delta^{\epsilon}_{\sigma}(\bar{\sigma}) \in \mathbb{R}$ is smooth. This implies[30] that for each fixed $\epsilon > 0$ and each $B \in \mathcal{B}'$ the function $B^{(\epsilon)} : \Sigma \to \mathfrak{t}$ given by $B^{(\epsilon)}(\sigma) = B(\delta^{\epsilon}_{\sigma})$ for all $\sigma \in \Sigma$ (cf. Convention 3 above) is smooth. Consequently, the function $\check{B}^{(\epsilon)} : \Sigma \to \mathfrak{t}$ given by $\check{B}^{(\epsilon)} = B^{(\epsilon)} - \int B^{(\epsilon)} d\mu_{\mathbf{g}}$ is a well-defined element of $\check{\mathcal{B}}$. (For $\check{B} \in \check{\mathcal{B}}' \subset \mathcal{B}'$ we will simply write $\check{B}^{(\epsilon)}$ instead of $\check{\check{B}}^{(\epsilon)}$.)

For fixed $y \in I$, $b \in \mathfrak{t}$, $s \in (0,1)$, and $\epsilon \in (0, \epsilon(s))$ we could now introduce the function $J^{(s,\epsilon)}_{b,y}(L) : \mathcal{E}' \to \mathbb{R}$ by

$$J^{(s,\epsilon)}_{b,y}(L)(\check{B}_1, \check{B}_2) := I^{(s,\epsilon)}_{rig}(L)(\star d\check{B}^{(\epsilon)}_1, \check{B}^{(\epsilon)}_2 + b) \cdot \exp(-2\pi i k \langle y, \check{B}^{(\epsilon)}_2(\sigma_0) + b \rangle) \times \operatorname{Det}_{rig}(\check{B}^{(\epsilon)}_2 + b) 1_{\mathcal{B}_{reg}}(\check{B}^{(\epsilon)}_2 + b) \quad (3.19)$$

for all $(\check{B}_1, \check{B}_2) \in (\check{\mathcal{B}})' \times (\check{\mathcal{B}})' \cong \mathcal{E}'$.

However, the last two factors $\operatorname{Det}_{rig}(\check{B}^{(\epsilon)}_2 + b)$ and $1_{\mathcal{B}_{reg}}(\check{B}^{(\epsilon)}_2 + b)$ are problematic since neither of these two factors (considered as functions $\mathcal{E}' \to \mathbb{R}$) is an element of $(\mathcal{E})$.

This is why, in addition to "point smearing", we will use an additional regularization and introduce regularized versions $1^{(n)}_{\mathcal{B}_{reg}} : \mathcal{B} \to \mathbb{R}$ and $\operatorname{Det}^{(n)}_{rig} : \mathcal{B} \to \mathbb{R}$, $n \in \mathbb{N}$, of $1_{\mathcal{B}_{reg}}$ and Det_{rig}. There are several ways to do this. In Appendix B we explain one possible regularization. The following definitions, results, and conjectures refer to the choice of $1^{(n)}_{\mathcal{B}_{reg}} : \mathcal{B} \to \mathbb{R}$ and $\operatorname{Det}^{(n)}_{rig} : \mathcal{B} \to \mathbb{R}$ of Appendix B.

Let $y \in I$, $b \in \mathfrak{t}$, $s \in (0,1)$, $\epsilon \in (0, \epsilon(s))$, and $n \in \mathbb{N}$ be fixed. We introduce the function $J^{(s,\epsilon,n)}_{b,y}(L) : \mathcal{E}' \to \mathbb{R}$ by

$$J^{(s,\epsilon,n)}_{b,y}(L)(\check{B}_1, \check{B}_2) := I^{(s,\epsilon)}_{rig}(L)(\star d\check{B}^{(\epsilon)}_1, \check{B}^{(\epsilon)}_2 + b) \cdot \exp(-2\pi i k \langle y, \check{B}^{(\epsilon)}_2(\sigma_0) + b \rangle) \times \operatorname{Det}^{(n)}_{rig}(\check{B}^{(\epsilon)}_2 + b) 1^{(n)}_{\mathcal{B}_{reg}}(\check{B}^{(\epsilon)}_2 + b) \quad (3.20)$$

for all $(\check{B}_1, \check{B}_2) \in (\check{\mathcal{B}})' \times (\check{\mathcal{B}})' \cong \mathcal{E}'$.

Proposition 3.3 *For all $b \in \mathfrak{t}$, $y \in I$, $s \in (0,1)$, $\epsilon \in (0, \epsilon(s))$, and $n \in \mathbb{N}$ we have*

$$J^{(s,\epsilon,n)}_{b,y}(L) \in (\mathcal{E})$$

Consequently, the expression $\Psi\big(J^{(s,\epsilon,n)}_{b,y}(L)\big)$ is well-defined.

After these preparations we can finally write down a rigorous version of the heuristic expression on the RHS of Eq. (2.44) above:

$$\operatorname{WLO}_{rig}(L) := \lim_{n\to\infty} \lim_{s\to 0} \lim_{\epsilon\to 0} \sum\nolimits_{y\in I} \int_{\sim} \Psi\big(J^{(s,\epsilon,n)}_{b,y}(L)\big) db \quad (3.21)$$

[30] In the special case $B = f \cdot d\mu_{\mathbf{g}}$ where $f : \Sigma \to \mathfrak{t}$ is continuous the smoothness of $B^{(\epsilon)} : \Sigma \to \mathfrak{t}$ follows easily from the assumption that $\Sigma \times \Sigma \ni (\sigma, \bar{\sigma}) \mapsto \delta^{\epsilon}_{\sigma}(\bar{\sigma}) \in \mathbb{R}$ is smooth. Moreover, the smoothness of $B^{(\epsilon)}$ follows also if B is any derivative of a distribution of the form $f \cdot d\mu_{\mathbf{g}}$ with $f \in C^0(\Sigma, \mathfrak{t})$. This covers already the general situation since, by a well-known theorem, every distribution $D \in \mathcal{D}'(\Sigma)$ can be written as a linear combination of derivatives of distributions of the form $f \cdot d\mu_{\mathbf{g}}$ where f is a continuous function $\Sigma \to \mathbb{R}$ and this result can immediately be generalized to the case of $\mathfrak{t}$-valued functions and distributions

where[31] $\int_\sim \cdots db$ is given by

$$\int_\sim f\, db = \lim_{T\to\infty} \tfrac{1}{(2T)^d} \int_{[-T,T]^d} f\, db$$

for any measurable bounded function $f : \mathfrak{t} \to \mathbb{R}$. Here $d = \dim(\mathfrak{t})$ and we have identified $\mathfrak{t}$ with $\mathbb{R}^d$ using any fixed orthonormal basis $(e_i)_{i\le d}$ of $\mathfrak{t}$.

Conjecture 1 $\mathrm{WLO}_{rig}(L)$ *is well-defined. In particular, all limits involved exist.*

If Conjecture 1 above is correct then in view of the semi-rigorous computations in [10, 6] (and the rigorous computations in [12, 13]) one naturally arrives at the following conjecture:

Conjecture 2 *Assume that $k > c_{\mathfrak{g}}$ where $c_{\mathfrak{g}} \in \mathbb{N}$ is the dual Coxeter number of $\mathfrak{g}$ (cf. Appendix A). Then we have for every L fulfilling Assumption 1 above*

$$\mathrm{WLO}_{rig}(L) \sim |L| \tag{3.22}$$

where $\sim$ denotes equality up to a multiplicative constant $C = C(G,k)$ and where $|\cdot|$ is the shadow invariant for $M = S^2 \times S^1$ associated to the pair $(\mathfrak{g}, k)$, cf. Appendix A for the definitions and concrete formulas (cf., in particular, Eq. A.10).

Remark 3.3 *i) As the notation $C = C(G,k)$ suggests the constant C referred to above is allowed to depend on G and k but will be independent of L. It will also be independent of the particular choice of the orthonormal basis $(e_i)_i$ of $\mathfrak{t}$ and the Dirac families $\{\delta^\epsilon_\sigma \mid \epsilon > 0, \sigma \in \Sigma\}$ and $\{\delta^\epsilon_t \mid \epsilon > 0, t \in S^1\}$ above. Finally, it will be independent of the particular choice of the auxiliary Riemannian metric $\mathbf{g}$ (as long as $\mathbf{g}$ fulfills Assumption 2 above).*

ii) Obviously we cannot expect Eq. (3.22) to hold with "$\sim$" replaced by "$=$" since in Sec. 2 we have omitted several multiplicative constants. Moreover, Eq. (2.44) contains "$\sim$" as well.

iii) In the standard literature the shadow invariant $|\cdot|$ associated to the pair $(\mathfrak{g}, k)$ is only defined when $k > c_{\mathfrak{g}}$. It can easily be generalized in a natural way so that it includes the situation $k \le c_{\mathfrak{g}}$ but it turns out that the so defined generalization of $|\cdot|$ vanishes for $k < c_{\mathfrak{g}}$ and is essentially trivial for $k = c_{\mathfrak{g}}$. We expect that the same applies to $\mathrm{WLO}_{rig}(L)$.

Remark 3.4 *Recall from Remark 2.2 above that I expect that both Conjecture 1 and Conjecture 2 can be generalized to the situation where L is a certain type of torus ribbon knot in $S^2 \times S^1$. In particular, it is very likely that using the approach above and one can obtain a rigorous continuum analogue of Theorem 5.7 in [14].*

4 Discussion & Outlook

4.1 Open Questions

Question 1 *Are Conjectures 1–2 above indeed true?*

If the answer to Question 1 is "yes", then one arrives naturally at the following question:

[31] recall that db is the normalized Lebesgue measure on $\mathcal{B}_c \cong \mathfrak{t}$. We expect that the function $\mathcal{B}_c \ni b \mapsto \Psi\big(J^{(s,\epsilon,n)}_{b,y}(L)\big) \in \mathbb{R}$ appearing in Eq. (3.21) is periodic. This is why instead of using the proper Lebesgue integral $\int \cdots db$ we use the "mean value" $\int_\sim \cdots db$

Question 2 *Can Assumption 1 be dropped? In other words: will the more or less straightforward*[32] *generalizations of Conjectures 1–2 to the case of generic*[33] *ribbon links also be true?*

Before one studies Question 2 on a rigorous level it is reasonable to consider this issue first on an informal level.

Apart from Questions 1 and 2, which are obviously the main questions, also the following two questions are of interest:

Question 3 *Can Assumption 2 be dropped? If not, then is there a deeper reason why we have to make such an assumption?*

Question 4 *Is it possible to find regularized versions* $1^{(n)}_{\mathcal{B}_{reg}}$ *and* $\mathrm{Det}^{(n)}_{rig}$, $n \in \mathbb{N}$, *of* $1_{\mathcal{B}_{reg}} : \mathcal{B} \to \mathbb{R}$ *and* $\mathrm{Det}_{rig} : \mathcal{B} \to \mathbb{R}$ *which are more natural than the ones given in Appendix B?*

4.2 Comparison with the simplicial approach

As mentioned in Sec. 1 there is an alternative approach for making rigorous sense of the RHS of Eq. (2.7), namely the "simplicial approach" developed in [12, 13]. The simplicial approach is essentially elementary[34] and, as a result, rigorous proofs can be obtained more easily than within the continuum approach of the present paper. Moreover, the simplicial approach is probably better suited for the kind of applications we have in mind (cf. "Problem 3" in the Introduction in [12]).

In spite of this it is still important to study and elaborate the rigorous continuum approach of the present paper. The following list should make clear:

- The continuum approach allows us to avoid the transition to BF-theory, which is apparently necessary in the simplicial apprroach.
- In the simplicial approach in [12, 13] there is one issue which is not totally understood. In the rigorous realization of $\mathrm{Det}_{FP}(B)$ in [12, 13] we need to include an 1/2-exponent in order to get the correct result. At the moment we only have a rather vague justification for this inclusion (cf. Appendix D in [13]). By contrast, in the rigorous continuum approach of the present paper this issue does not play a role.
- In view of Conjecture 1 and Conjecture 2, for the special type of (ribbon) links L fulfilling Assumption 1 the approach of the present paper should lead to the same explicit expressions as the approach in [12, 13]. However, for general ribbon links L this is most probably not the case. It may well turn out that for general ribbon links only the continuum approach will give us the correct expressions for the WLOs while the simplicial approach in its original form does not.
- Finally, the rigorous version of continuum approach is closer to heuristic computations in [15]. So if the project in [15] can be carried out successfully, it will almost certainly be clear that by adapting the approach of the present paper one can obtain a rigorous treatment within the framework of White Noise Analysis.

[32] Recall that when Assumption 1 is dropped we need to modify Assumption 2 (cf. Remark 2.6 in Sec. 2.5) and some of the constructions & definitions in Sec. 3.3. Moreover, we need to give a (heuristic) derivation/justification for formula (2.39) in Sec. 2.6 also in the case of general ribbon links L.

[33] in fact, we expect that the class of ribbon links for which our approach is applicable cannot be the class of general ribbon links $L = (R_1, R_2, \ldots, R_m)$. All "singular" twists of the ribbons R_i, $i \le m$, must probably be excluded. One sufficient condition on L which excludes such singular twists is that each R^i_Σ is a local diffeomorphism. Of course, crossings and self-crossing of the projected ribbons R^i_Σ, $i \le m$ (and certain "regular" twists) are nor excluded by this condition

[34] with the exception of some general results from Lie theory

A Appendix: Turaev's shadow invariant

Let us briefly recall the definition of Turaev's shadow invariant in the situation relevant for us, i.e. for the base manifold $M = \Sigma \times S^1$ with $\Sigma = S^2$.

A.1 Lie theoretic notation

Let G, T, $\mathfrak{g}$, $\mathfrak{t}$, $\langle \cdot, \cdot \rangle$, and $\mathfrak{k}$ be as in Sec. 2 above. Using the scalar product $\langle \cdot, \cdot \rangle$ we can make the identification $\mathfrak{t} \cong \mathfrak{t}^*$. Let us now fix a Weyl chamber $\mathcal{C} \subset \mathfrak{t}$ and introduce the following notation:

- $\mathcal{R} \subset \mathfrak{t}^*$: the set of real roots associated to $(\mathfrak{g}, \mathfrak{t})$
- $\mathcal{R}_+ \subset \mathcal{R}$: the set of positive (real) roots corresponding to $\mathcal{C}$
- ρ: half sum of positive roots ("Weyl vector")
- θ: unique long root in the Weyl chamber $\mathcal{C}$.
- $c_{\mathfrak{g}} = 1 + \langle \theta, \rho \rangle$: the dual Coxeter number of $\mathfrak{g}$.
- $I \subset \mathfrak{t}$: the kernel of $\exp_{|\mathfrak{t}} : \mathfrak{t} \to T$. We remark that from the assumption that G is simply-connected it follows that I coincides with the lattice Γ which is generated by the set of real coroots associated to $(\mathfrak{g}, \mathfrak{t})$.
- $\Lambda \subset \mathfrak{t}^* (\cong \mathfrak{t})$: the *real* weight lattice associated to $(\mathfrak{g}, \mathfrak{t})$, i.e. Λ is the lattice which is dual to $\Gamma = I$.
- $\Lambda_+ \subset \Lambda$: the set of dominant weights corresponding to $\mathcal{C}$, i.e. $\Lambda_+ := \bar{\mathcal{C}} \cap \Lambda$
- $\Lambda_+^k \subset \Lambda$, $k \in \mathbb{N}$: the subset of Λ_+ given by $\Lambda_+^k := \{\lambda \in \Lambda_+ \mid \langle \lambda, \theta \rangle \leq k - c_{\mathfrak{g}}\}$ (the "set of dominant weights which are integrable at level $l := k - c_{\mathfrak{g}}$").
- $\mathcal{W} \subset \mathrm{GL}(\mathfrak{t})$: the Weyl group of the pair $(\mathfrak{g}, \mathfrak{t})$
- $\mathcal{W}_{\text{aff}} \subset \mathrm{Aff}(\mathfrak{t})$: the "affine Weyl group of $(\mathfrak{g}, \mathfrak{t})$", i.e. the subgroup of $\mathrm{Aff}(\mathfrak{t})$ generated by $\mathcal{W}$ and the set of translations $\{\tau_x \mid x \in \Gamma\}$ where $\tau_x : \mathfrak{t} \ni b \mapsto b + x \in \mathfrak{t}$.
- $\mathcal{W}_k \subset \mathrm{Aff}(\mathfrak{t})$, $k \in \mathbb{N}$: the subgroup of $\mathrm{Aff}(\mathfrak{t})$ given by $\{\psi_k \circ \sigma \circ \psi_k^{-1} \mid \sigma \in \mathcal{W}_{\text{aff}}\}$ where $\psi_k : \mathfrak{t} \in b \mapsto b \cdot k - \rho \in \mathfrak{t}$ (the "quantum Weyl group corresponding to the level $l := k - c_{\mathfrak{g}}$")

The following formulas are used in Sec. 2.5 above and Appendix B below. For $b \in \mathfrak{t}$ we have

$$b \in \mathfrak{t}_{reg} \quad \Leftrightarrow \quad [\forall \alpha \in \mathcal{R}_+ : \ \alpha(b) \notin \mathbb{Z}] \tag{A.1}$$

and

$$\begin{aligned} \det(\mathrm{id}_{\mathfrak{k}} - \exp(\mathrm{ad}(b))_{|\mathfrak{k}}) &= \prod\nolimits_{\alpha \in \mathcal{R}} (1 - e^{2\pi i \alpha(b)}) \\ &= \prod\nolimits_{\alpha \in \mathcal{R}_+} (1 - e^{2\pi i \alpha(b)})(1 - e^{-2\pi i \alpha(b)}) = \prod\nolimits_{\alpha \in \mathcal{R}_+} 4 \sin^2(\pi \alpha(b)) \end{aligned} \tag{A.2}$$

A.2 The shadow invariant

Let $L = (l_1, l_2, \ldots, l_m)$, $m \in \mathbb{N}$, be a framed link in $M = \Sigma \times S^1$. For simplicity we will assume that each l_i, $i \leq m$ is equipped with a "horizontal" framing[35]. Let $V(L)$ denote the set of

[35] here we use the terminology of Remark 4.5 in [12]. We remark that in the special case when L is the framed link which is induced by a ribbon link $L = L_{ribb}$ fulfilling Assumption 1 in Sec. 2.4 then each l_i will automatically be equipped with a horizontal framing

points $p \in \Sigma$ where the loops l^i_Σ, $i \le m$, cross themselves or each other (the "crossing points") and $E(L)$ the set of curves in Σ into which the loops $l^1_\Sigma, l^2_\Sigma, \ldots, l^m_\Sigma$ are decomposed when being "cut" in the points of $V(L)$. We assume that there are only finitely many connected components $Y_0, Y_1, Y_2, \ldots, Y_{m'}$, $m' \in \mathbb{N}$ ("faces") of $\Sigma \backslash (\bigcup_i \operatorname{arc}(l^i_\Sigma))$ and set

$$F(L) := \{Y_0, Y_1, Y_2, \ldots, Y_{m'}\}.$$

As explained in [27] one can associate in a natural way a half integer $\operatorname{gleam}(Y) \in \frac{1}{2}\mathbb{Z}$, called "gleam" of Y, to each face $Y \in F(L)$. In the special case where the link L is a framed link which is induced by a ribbon link $L = L_{ribb}$ fulfilling Assumption 1 in Sec. 2.4 we have the explicit formula

$$\operatorname{gleam}(Y) = \sum\nolimits_{i \text{ with } \operatorname{arc}(l^i_\Sigma) \subset \partial Y} \operatorname{wind}(l^i_{S^1}) \cdot \operatorname{sgn}(Y; l^i_\Sigma) \in \mathbb{Z} \tag{A.3}$$

where $\operatorname{wind}(l^i_{S^1})$ is the winding number of the loop $l^i_{S^1}$ and where $\operatorname{sgn}(Y; l^i_\Sigma)$ is given by

$$\operatorname{sgn}(Y; l^i_\Sigma) := \begin{cases} 1 & \text{if } Y \subset Z^+_i \\ -1 & \text{if } Y \subset Z^-_i \end{cases} \tag{A.4}$$

Here Z^+_i (resp. Z^-_i) is the unique connected component Z of $\Sigma \backslash \operatorname{arc}(l^i_\Sigma)$ such that l^i_Σ runs around Z in the "positive" (resp. "negative") direction.

Assume that each loop l_i in the link L is equipped with a "color" ρ_i, i.e. a finite-dimensional complex representation of G. By $\gamma_i \in \Lambda_+$ we denote the highest weight of ρ_i and set $\gamma(e) := \gamma_i$ for each $e \in E(L)$ where $i \le n$ denotes the unique index such that $\operatorname{arc}(e) \subset \operatorname{arc}(l_i)$. Finally, let $col(L)$ be the set of all mappings $\varphi : \{Y_0, Y_1, Y_2, \ldots, Y_{m'}\} \to \Lambda^k_+$ ("area colorings").

We can now define the "shadow invariant" $|L|$ of the (colored and "horizontally framed") link L associated to the pair $(\mathfrak{g}, k)$ by

$$|L| := \sum\nolimits_{\varphi \in col(L)} |L|^\varphi_1 \, |L|^\varphi_2 \, |L|^\varphi_3 \, |L|^\varphi_4 \tag{A.5}$$

with

$$|L|^\varphi_1 = \prod\nolimits_{Y \in F(L)} \dim(\varphi(Y))^{\chi(Y)} \tag{A.6a}$$

$$|L|^\varphi_2 = \prod\nolimits_{Y \in F(L)} \exp(\tfrac{\pi i}{k} \langle \varphi(Y), \varphi(Y) + 2\rho \rangle)^{\operatorname{gleam}(Y)} \tag{A.6b}$$

$$|L|^\varphi_3 = \prod\nolimits_{e \in E_*(L)} N^{\varphi(Y^-_e)}_{\gamma(e)\varphi(Y^+_e)} \tag{A.6c}$$

$$|L|^\varphi_4 = (\prod\nolimits_{e \in E(L) \backslash E_*(L)} S(e, \varphi)) \times (\prod\nolimits_{x \in V(L)} T(x, \varphi)) \tag{A.6d}$$

Here Y^+_e (resp. Y^-_e) denotes the unique face Y such that $\operatorname{arc}(e) \subset \partial Y$ and, additionally, the orientation on $\operatorname{arc}(e)$ described above coincides with (resp. is opposite to) the orientation which is obtained by restricting the orientation on ∂Y to e. Moreover, we have set (for $\lambda, \mu, \nu \in \Lambda^k_+$)

$$\dim(\lambda) := \prod\nolimits_{\alpha \in \mathcal{R}_+} \frac{\sin \frac{\pi \langle \lambda + \rho, \alpha \rangle}{k}}{\sin \frac{\pi \langle \rho, \alpha \rangle}{k}} \tag{A.7}$$

$$N^\lambda_{\mu\nu} := \sum\nolimits_{\tau \in \mathcal{W}_k} \operatorname{sgn}(\tau) m_\mu(\nu - \tau(\lambda)) \tag{A.8}$$

where $m_\mu(\beta)$ is the multiplicity of the weight β in the unique (up to equivalence) irreducible representation ρ_μ with highest weight μ and $\mathcal{W}_k$ is as above. $E_*(L)$ is a suitable subset of $E(L)$ (cf. the notion of "circle-1-strata" in Chap. X, Sec. 1.2 in [28]).

The explicit expression for the factors $T(x, \varphi)$ appearing in $|L|_4^{\varphi}$ above involves the so-called "quantum 6j-symbols" (cf. Chap. X, Sec. 1.2 in [28]) associated to the quantum group $U_q(\mathfrak{g}_{\mathbb{C}})$ where q is the root of unity

$$q := \exp(\tfrac{2\pi i}{k}) \tag{A.9}$$

We omit the explicit formulae for $T(x, \varphi)$ and $S(e, \varphi)$ since they irrelevant for our purposes. Indeed, if L is the framed link which is induced by a ribbon link $L = L_{ribb}$ fulfilling Assumption 1 in Sec. 2.4 the set $V(L)$ is empty and the set $E_*(L)$ coincides with $E(L)$, so Eq. (A.5) then reduces to

$$|L| = \sum_{\varphi \in col(L)} \Bigl(\prod_{i=1}^{m} N_{\gamma(l_i)\varphi(Y_i^+)}^{\varphi(Y_i^-)} \Bigr) \Bigl(\prod_{Y \in F(L)} \dim(\varphi(Y))^{\chi(Y)} \exp(\tfrac{\pi i}{k} \langle \varphi(Y), \varphi(Y) + 2\rho \rangle)^{\mathrm{gleam}(Y)} \Bigr) \tag{A.10}$$

where we have set $Y_i^{\pm} := Y_{l_\Sigma^i}^{\pm}$.

B Appendix: Construction of $1_{\mathcal{B}_{reg}}^{(n)}$ and $\mathrm{Det}_{rig}^{(n)}$

We will now describe the regularized versions $1_{\mathcal{B}_{reg}}^{(n)} : \mathcal{B} \to \mathbb{R}$ and $\mathrm{Det}_{rig}^{(n)} : \mathcal{B} \to \mathbb{R}$, $n \in \mathbb{N}$, of the functions $1_{\mathcal{B}_{reg}}$ and Det_{rig} which we used in Sec. 3.4 above.

- Let $\triangle_0$ be a fixed finite triangulation of Σ which is "compatible" with L in the sense that $\mathrm{arc}(L_\Sigma)$ is contained in the 1-skeleton of $\triangle_0$. For each $n \in \mathbb{N}$ let $\triangle_n$ be the barycentric sub division of $\triangle_{n-1}$. We denote by $\mathcal{F}_2(\triangle_n)$ the set of 2-faces of $\triangle_n$. For $B \in \mathcal{B}$ and $F \in \mathcal{F}_2(\triangle_n)$ let $B(F)$ be the "mean value" of B on F, i.e.

$$B(F) := \frac{\int_F B d\mu_{\mathbf{g}}}{\int_F 1 d\mu_{\mathbf{g}}} \in \mathfrak{t} \tag{B.1}$$

- We approximate the indicator function $1_{\mathfrak{t}_{reg}} : \mathfrak{t} \to [0,1]$ by a suitable sequence of trigonometric polynomials[36] $(1_{\mathfrak{t}_{reg}}^{(n)})_{n \in \mathbb{N}}$ such that for all $n \in \mathbb{N}$ we have

$$1_{\mathfrak{t}_{reg}}^{(n)}(b) = 0 \quad \text{for all } b \in \mathfrak{t}_{sing} := \mathfrak{t} \backslash \mathfrak{t}_{reg}, \text{ and}$$
$$|1_{\mathfrak{t}_{reg}}^{(n)}(b) - 1| \leq 1/N_n^2 \quad \text{for all } b \in \mathfrak{t} \text{ outside the } 1/n\text{-neighborhood of } \mathfrak{t}_{sing}$$

  where we have set $N_n := \#\mathcal{F}_2(\triangle_n)$.

  For each $n \in \mathbb{N}$ we then introduce $1_{\mathcal{B}_{reg}}^{(n)} : \mathcal{B} \to \mathbb{R}$ by

$$1_{\mathcal{B}_{reg}}^{(n)}(B) = \prod_{F \in \mathcal{F}_2(\triangle_n)} 1_{\mathfrak{t}_{reg}}^{(n)}(B(F)) \quad \forall B \in \mathcal{B}$$

- Recall from Sec. 2.5 that for $B \in \mathcal{B}_{reg}$ we have

$$\mathrm{Det}_{rig,\alpha}(B) = \exp\Bigl(\int_\Sigma \bigl[\log\bigl(2 \sin(\pi \alpha(B(\sigma)))\bigr) \tfrac{1}{4\pi} R_{\mathbf{g}}(\sigma) \bigr] d\mu_{\mathbf{g}}(\sigma) \Bigr)$$

[36] in order to see that such a sequence always exist we first choose, for each fixed $n \in \mathbb{N}$, a smooth 1-periodic function $\psi^{(n)} : \mathbb{R} \to \mathbb{R}$ such that $\psi^{(n)}(x) = 0$ for all $x \in \mathbb{Z}$ and $\psi^{(n)}(x) = 1$ for all x outside the $\frac{C}{n}$-neighborhood of $\mathbb{Z} \subset \mathbb{R}$ for some fixed $C > 0$. Since $\psi^{(n)}$ is a smooth periodic function its Fourier series converges uniformly. Accordingly, for every fixed $\epsilon = \epsilon_n > 0$ we can find a (1-periodic) trigonometric polynomial $p^{(n)}$ such that $\|\psi^{(n)} - p^{(n)}\|_\infty < \epsilon$ and, consequently, $\|\psi^{(n)} - \bar{p}^{(n)}\|_\infty < 2\epsilon$ where $\bar{p}^{(n)} := p^{(n)} - p^{(n)}(0)$. (Clearly, $\bar{p}^{(n)}(x) = 0$ for all $x \in \mathbb{Z}$.) In view of the identity $1_{\mathfrak{t}_{reg}}(b) = \prod_{\alpha \in \mathcal{R}_+} 1_{\mathbb{R} \backslash \mathbb{Z}}(\alpha(b))$, cf. Eq. (A.1) above, we now define $1_{\mathfrak{t}_{reg}}^{(n)}$ by $1_{\mathfrak{t}_{reg}}^{(n)}(b) := \prod_{\alpha \in \mathcal{R}_+} \bar{p}^{(n)}(\alpha(b))$ for all $b \in \mathfrak{t}$. Clearly, if $C > 0$ was chosen small enough and for every $n \in \mathbb{N}$ the number $\epsilon = \epsilon_n$ was chosen small enough we obtain a family $(1_{\mathfrak{t}_{reg}}^{(n)})_{n \in \mathbb{N}}$ with the desired properties

where $\log : \mathbb{R}\backslash\{0\} \to \mathbb{C}$ is the restriction to $\mathbb{R}\backslash\{0\}$ of the principal branch of the complex logarithm, i.e. is given by

$$\log(x) = \ln(|x|) + \pi i H(-x) \quad \forall x \in \mathbb{R}\backslash\{0\}$$

where $H(x) = (1 + \operatorname{sgn}(x))/2$ is the Heaviside function.

For $B \in \mathcal{B}$ and $n \in \mathbb{N}$ we now define the "step function" $B^{(n)} : \Sigma \to \mathfrak{t}$ by

$$B^{(n)} = \sum\nolimits_{F \in \mathcal{F}_2(\triangle_n)} B(F) \cdot 1_F \tag{B.2}$$

where $B(F)$ is as above. Moreover, for fixed $n \in \mathbb{N}$ we now replace $\exp(x)$, $x \in \mathbb{R}$, by the nth Taylor polynomial $\exp^{(n)}(x) = \sum_{k=0}^{n} \frac{x^k}{k!}$ and $\log(x)$ by $\log^{(n)}(x)$ where $(\log^{(n)})_{n\in\mathbb{N}}$ is any fixed sequence of polynomial functions $\log^{(n)}(x) : I \to \mathbb{C}$ which converges uniformly to log on every compact subinterval of $I := [-2, 2]\backslash\{0\}$.

After these preparations we set for each $B \in \mathcal{B}$.

$$\mathrm{Det}^{(n)}_{rig,\alpha}(B) := \exp^{(n)}\left(\int_\Sigma \left[\log^{(n)}(2\sin(\pi\alpha(B^{(n)}(\sigma)))) \tfrac{1}{4\pi} R_{\mathbf{g}}(\sigma)\right] d\mu_{\mathbf{g}}(\sigma) \right) \tag{B.3a}$$

and define $\mathrm{Det}^{(n)}_{rig} : \mathcal{B} \to \mathbb{R}$, $n \in \mathbb{N}$, by

$$\mathrm{Det}^{(n)}_{rig}(B) := \prod\nolimits_{\alpha \in \mathcal{R}_+} \mathrm{Det}^{(n)}_{rig,\alpha}(B) \quad \forall B \in \mathcal{B} \tag{B.3b}$$

Remark B.1 i) After carrying out the $\epsilon \to 0$ and the $s \to 0$-limits on the RHS of Eq. (3.21) step functions of the type $B = \sum_{i=1}^r b_i 1_{Y_i}$ appear in the calculations, cf. the paragraph after Eq. (2.28) in Sec. 2.5. It is useful (and straightforward) to generalize the definitions of $1^{(n)}_{\mathcal{B}_{reg}}(B)$ and $\mathrm{Det}^{(n)}_{rig}(B)$ to such functions B.

ii) In order to justify that $(1^{(n)}_{\mathcal{B}_{reg}})_{n\in\mathbb{N}}$ is indeed a regularization of $1_{\mathcal{B}_{reg}}$ observe that if $B \in \mathcal{B}_{reg}$ then

$$\lim_{n\to\infty} 1^{(n)}_{\mathcal{B}_{reg}}(B) = 1$$

On the other hand, we do not necessarily have $\lim_{n\to\infty} 1^{(n)}_{\mathcal{B}_{reg}}(B) = 0$ if $B \notin \mathcal{B}_{reg}$. However, if B is a step function of the type $B = \sum_{i=1}^r b_i 1_{Y_i}$ as in Eq. (2.28) in Sec. 2.5 (which is the only case which will play a role in our computations after the $\epsilon \to 0$ and the $s \to 0$-limits on the RHS of Eq. (3.21) have been taken) then we do have[37]

$$\lim_{n\to\infty} 1^{(n)}_{\mathcal{B}_{reg}}(B) = \begin{cases} 1 & \text{if } B \text{ takes only values in } \mathfrak{t}_{reg} \\ 0 & \text{if } B \text{ takes at least one value in } \mathfrak{t}_{sing} \end{cases}$$

where $1^{(n)}_{\mathcal{B}_{reg}}(B)$ is the aforementioned generalization.

iii) In order to justify that $(\mathrm{Det}^{(n)}_{rig})_{n\in\mathbb{N}}$ is a regularization of $\mathrm{Det}_{rig}(B)$ we will now verify that for all $B \in \mathcal{B}_{reg}$ we have indeed

$$\lim_{n\to\infty} \mathrm{Det}^{(n)}_{rig}(B) = \mathrm{Det}_{rig}(B) \tag{B.4}$$

Observe first that since $B \in \mathcal{B}_{reg}$ we have $B(\sigma) \in \mathfrak{t}_{reg}$ and therefore (cf. Eq. (A.1)) $\sin(\pi\alpha(B(\sigma)) \neq 0$ for all $\sigma \in \Sigma$. Now let $\mathcal{S}_n$ be the 1-skeleton of $\triangle_n$ and let $\mathcal{N} = \bigcup_{n\in\mathbb{N}} \mathcal{S}_n$.

[37]Here we use the condition mentioned above that the triangulation $\triangle_0$ of Σ and therefore also all barycentric subdivisions $\triangle_n$ of $\triangle_0$ are compatible with L in the sense above

Then for all $\sigma \in \Sigma$ which do not lie in $\mathcal{N}$ we have $\lim_{n\to\infty} B^{(n)}(\sigma) = B(\sigma)$ and therefore also $\sin(\pi\alpha(B^{(n)}(\sigma)) \neq 0$ if $n \in \mathbb{N}$ is sufficiently large. According to the choice of $\log^{(n)}$ we therefore obtain

$$\lim_{n\to\infty} \log^{(n)}(2\sin(\pi\alpha(B^{(n)}(\sigma))) = \log(2\sin(\pi\alpha(B(\sigma))) \tag{B.5}$$

Since $\mathcal{N}$ is a $\mu_{\mathbf{g}}$-zero set Eq. (B.5) holds for $\mu_{\mathbf{g}}$-almost all $\sigma \in \Sigma$. From Eqs. (B.3) it now easily follows that Eq. (B.4) is indeed fulfilled.

Finally, observe that for step functions $B = \sum_{i=1}^{r} b_i 1_{Y_i}$ of the type mentioned above which satisfy $1^{(n)}_{\mathcal{B}_{reg}}(B) \neq 0$ for sufficiently large n we have[38] again Eq. (B.4) (and, in fact, even $\mathrm{Det}^{(n)}_{rig}(B) = \mathrm{Det}_{rig}(B)$ for sufficiently large n).

I want to emphasize once more that the regularized versions $1^{(n)}_{\mathcal{B}_{reg}} : \mathcal{B} \to \mathbb{R}$ and $\mathrm{Det}^{(n)}_{rig} : \mathcal{B} \to \mathbb{R}$ of the functions $1_{\mathcal{B}_{reg}}$ and Det_{rig} introduced above are probably not the best ones. It would be desirable to find a more elegant and more natural regularization (cf. Question 4 in Sec. 4 above). In particular, one should try to avoid the use of triangulations, which is clearly not in the spirit of a continuum approach.

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